Logic As A Tool For Building Theories

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Introduction

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Hence new perspective on the question:

$\boxed{\text{What logic is — and should be!}}$

The Plan

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Aims: not to put forward any philosophical theses, but to provide some materials and raise some questions.

Case Study I: Modal and Temporal Logic

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- More concretely: from ‘possible worlds’ and ‘accessibility’ to **states** and **transitions**. Systems of states evolving under discrete transitions turn up in a huge variety of situations. (Communications protocols, hardware circuits, software, nowadays biological and physical systems . . .). Modal and temporal logics are canonical formalisms for expressing and reasoning about properties of such systems.

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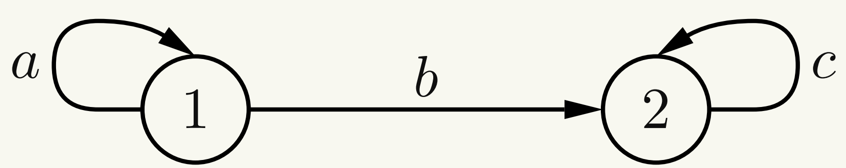
A transition system:



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- Huge expansion to cover real-time, probabilistic and hybrid systems, and of applications to include biological systems, security, networks, agent-based modelling, control systems etc.

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- A deep conceptual issue of logic in CS: the 'next 700 . . . problem'.

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Cf. André Weil: he compared finding the right definitions in algebraic number theory — which was like carving adamantine rock — to making definitions in the theory of uniform spaces (which he founded), which was like sculpting with snow.

Case Study II: λ-calculus

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Variables x, y, z, $\ldots$

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E.g.

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By orienting this equation, we get a 'dynamics' — β**-reduction**

$$(\lambda x.\, t)u \to t[u/x]$$

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  Historically, Curry extracted $\mathbf{Y}$ from an analysis of **Russell's Paradox**.

Simple Type System for $\times$, $\rightarrow$.

Variable

$$\frac{}{\Gamma, x : t \vdash x : T}$$

Product

$$\frac{\Gamma \vdash t : T \qquad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U} \qquad \frac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1 v : T} \qquad \frac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2 v : U}$$

Function

$$\frac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x.\, t : U \rightarrow T} \qquad \frac{\Gamma \vdash t : U \rightarrow T \qquad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$$

Natural Deduction System for $\wedge$, $\supset$

Identity

$$\frac{}{\Gamma, A \vdash A}\ \text{Id}$$

Conjunction

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\text{-intro} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-elim-1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-elim-2}$$

Implication

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \supset\text{-intro} \qquad \frac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \supset\text{-elim}$$

Types and Curry-Howard Correspondence

If we equate

$$\begin{array}{ccc} \wedge & \equiv & \times \\ \supset & \equiv & \longrightarrow \end{array}$$

they are the same!

Developments

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- Paradoxes: not just biting the bullet — not bugs but features! Recursion, fixpoints, the creative uses of computationally specified infinite objects.

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A discussion of domain theory emphasizing conceptual aspects in my article in the Handbook of Philosophy of Information (ed. van Benthem and Adriaans, Elsevier 2008).

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- Game Semantics, full abstraction and full completeness. Again, see my article in the Handbook of Philosophy of Information (and, hopefully, forthcoming article in SEP).

Case Study III: Category Theory and Coalgebra

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  Examples: Cohomology, categorification, the microcosm principle.

Coalgebras

Category theory allows us to **dualize** our entire discussion of algebras to obtain a notion of **coalgebras of an endofunctor**. However, while algebras abstract a familiar set of notions (inductive data types, structural recursion), colagebras open up a new and rather unexpected territory, and provides an effective abstraction and mathematical theory for a central class of computational phenomena:

- Programming over **infinite data structures**: streams, lazy lists, infinite trees ...
- A novel notion of **coinduction**
- Modelling **state-based computations** of all kinds
- The key notion of **bisimulation equivalence** between processes.

Basic Concepts

Let $F : \mathcal{C} \longrightarrow \mathcal{C}$ be a functor.

An F**-coalgebra** is an arrow $\gamma : A \longrightarrow FA$ for some object A of $\mathcal{C}$. We say that A is the **carrier** of the coalgebra, while γ is the **behaviour map**.

An F**-coalgebra homomorphism** from $\gamma : A \longrightarrow FA$ to $\delta : B \longrightarrow FB$ is an arrow $h : A \longrightarrow B$ such that

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, "h"'] & FA \arrow[d, "Fh"] \\ B \arrow[r, "\delta"'] & FB \end{tikzcd}$$

F-coalgebras and their homomorphisms form a category $F\!-\!\mathbf{Coalg}$.

An F-coalgebra γ is **final** if for every F-coalgebra δ there is a unique homomorphism from δ to γ.

Proposition 1 *If a final F-coalgebra exists, it is unique up to isomorphism.*

Proposition 2 (Lambek Lemma) *If $\gamma : A \longrightarrow FA$ is final, it is an isomorphism*

Let A be a set of **actions**. A **labelled transition system over** A is a coalgebra for the functor

$$\mathsf{LT}_A : \mathbf{Set} \longrightarrow \mathbf{Set} :: X \mapsto \mathcal{P}_\mathsf{f}(A \times X).$$

Such a coalgebra

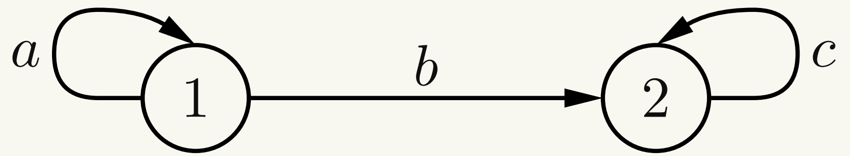
$$\gamma : S \longrightarrow \mathcal{P}_\mathsf{f}(A \times S)$$

can be understood operationally as follows:

- S is a set of **states**
- For each state $s \in S$, $\gamma(s)$ specifies the possible **transitions** from that state. We write $s \xrightarrow{a} s'$ if $(a, s') \in \gamma(s)$. We think of such a transition as consisting of the system performing the action a, and changing state from s to s'. Note that we regard actions as directly **observable**, while states are not.

Note that any labelled transition graph (or "state machine") with labels in A is a coalgebra for LT_A.

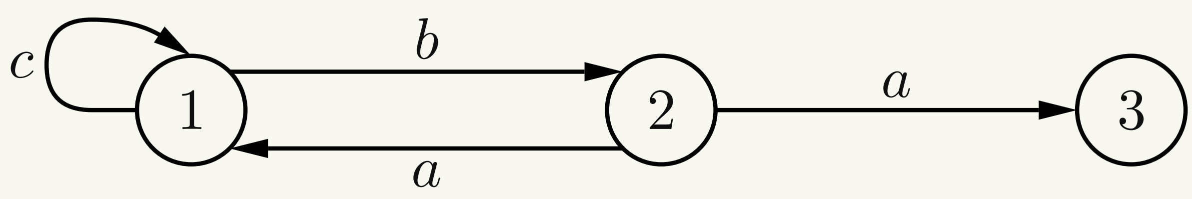
Examples 1.



This corresponds to the coalgebra $(\{1, 2\}, \gamma)$

$$\gamma : 1 \mapsto \{(a, 1), (b, 2)\}, \qquad 2 \mapsto \{(c, 2)\}$$

2.



$$1 \mapsto \{(b, 2), (c, 1)\}, \qquad 2 \mapsto \{(a, 1), (a, 3)\}, \qquad 3 \mapsto \varnothing$$

The key point is this.

Proposition 3 *For any set A of actions, there is a final LT_A-coalgebra $(\mathsf{Proc}_A, \mathsf{out})$.*

The Final Coalgebra

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We think of elements of the final coalgebra as **processes**. The final coalgebra provides a “universal semantics” for transition systems, which is “fully abstract” with respect to observable behaviour.

The key point is this.

Proposition 3 *For any set A of actions, there is a final LT_A-coalgebra $(\mathsf{Proc}_A, \mathsf{out})$.*

We think of elements of the final coalgebra as **processes**. The final coalgebra provides a “universal semantics” for transition systems, which is “fully abstract” with respect to observable behaviour.

All of this generalizes to a large class of endofunctors.

Some Remarks and Questions for P-B PoL

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  Examples: non-well-founded sets, even non-well-found proofs!

Final Remarks

Logic As A Tool For Building Theories

Computer Science theories of:

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Considerable potential beyond Computer Science: in physics, biology, cognitive and social sciences etc.

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Contrast: Philosophy of Physics vs. Philosophy of Logic and Mathematics.