Introduction

Logical and Mathematical Reasoning

Conclusion

# Formalization is Idealization

## Uncovering the subtle nuances of logical and mathematical practices

Benedikt Löwe & Thomas Müller

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# Motivation (1).

4,000 years of arithmetic
2,500 years of geometry $\longrightarrow$ **Set-theoretic foundations**
500 years of real analysis

| | |
|---|---|
| natural numbers | $\{0,1,2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ |
| lines | sets of points $g = \{\{\{x\},\{x,y\}\}\,;\, a \cdot x + b = y\}$ |
| real numbers | Dedekind cuts |

This formalization allows us to ask the following questions:

1. $7 \in 9$?
2. $g \cap \{U\,;\, U \text{ is an ultrafilter on } \mathbb{N}\} = \varnothing$?
3. $\mathbf{1}_{\mathbb{N}} = \mathbf{1}_{\mathbb{R}}$?

Paul Benacerraf, What numbers could not be?, *Philosophical Review* 74 (1965):47–73.

# Motivation (2).



**Example.**

Dieter Landers, Lothar Rogge, Nichtstandard Analysis, Springer 1994

When defining the non-standard real numbers ${}^*\mathbb{R}$ as an ultrapower $\mathrm{Ult}$ of $\mathbb{R}$ with an ultrafilter on $\omega$, you get an embedding $j$ from $\mathbb{R}$ to ${}^*\mathbb{R}$ by $x \mapsto$ the equivalence class of the constant sequence with value $x$. But $x \neq j(x)$ as a set-theoretic object.

Landers & Rogge define

$$ {}^*\mathbb{R} := \mathrm{Ult}\backslash\{C\,;\,\exists x \in \mathbb{R}(C = [\mathrm{const}_x])\} \cup \mathbb{R}. $$

**To sum up:** The formalization / representation in formal language is not identical to the concepts formalized.

# Motivation (3).

**We see:** Formalization has sociological effects: an accepted formalisation has normative force.

While the original ZFC set-up was pragmatically driven, after the arbitrary decision to use set-theoretic foundations and the community consensus on its axiomatization as ZFC, it becomes the source for ontological, epistemological and ideological arguments.

# Idealization and Abstraction (1). Terminology.

*An idealization is a deliberate simplification of something complicated with the objective of making it more tractable. [...] [There are] two general kinds of idealizations: so-called Aristotelian and Galilean idealizations.*
*Aristotelian idealization amounts to 'stripping away', in our imagination, all properties from a concrete object that we believe are not relevant to the problem at hand. This allows us to focus on a limited set of properties in isolation. [...]*
*Galilean idealizations are ones that involve deliberate distortions.*

*R. Frigg & S. Hartmann (2006), "Models in science", Stanford Encyclopedia*

*I intend to take [the] distinction between misrepresentation and mere omission as fundamental, and to suggest that we organize our terminology around it. On the regimentation of usage I am thus proposing, the term 'idealization' applies, first and foremost, to specific respects in which a given representation misrepresents, whereas the term 'abstraction' applies to mere omissions. [...] we should take idealization to require the assertion of a falsehood, and abstraction to involve the omission of a truth.*

*M.R. Jones (2005), "Idealization and abstraction: A framework", 174f.*

# Idealization and Abstraction (2). Definitions.

Following Jones (2005):

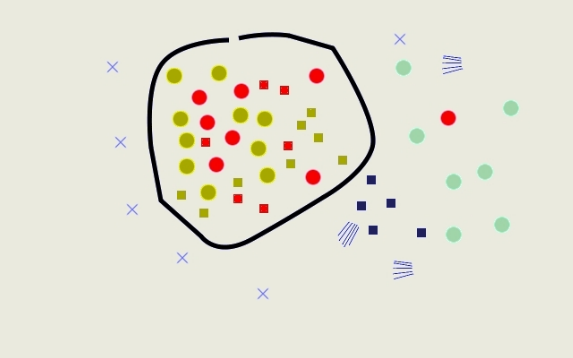
**Abstraction (a.k.a. Aristotelian idealization):**
Given a class of individuals, an *abstraction* is a concept under which all of the individuals fall.

**Idealization (a.k.a. Galilean idealization):**
Given a class of individuals, an *idealization* is a concept under which all of the individuals almost fall (in some pragmatically relevant sense), while at least one individual is excluded by the idealization.

Both are present in actual scientific practice.
In fact, "virtuous distortions" are everywhere in representation; even in scale models (Plato, *Sophistes*)

Van Fraassen (2008). *Scientific Representation: Paradoxes of Perspective.*

# Idealization and Abstraction (3). Examples.



**Abstraction.**
$(\texttt{colour} = \texttt{red} \lor \texttt{colour} = \texttt{green}) \land (\texttt{shape} = \texttt{circle} \lor \texttt{shape} = \texttt{square})$
(Abstraction typically enlarges the class.)

**Idealization.**
$\texttt{colour} = \texttt{gray} \land \texttt{shape} = \texttt{hexagon}$
(Idealization typically produces a smaller class.)

# Idealization and Abstraction (4). Examples.

*Linguistic theory is concerned primarily with an ideal speaker-listener, in a completely homogeneous speech-community, who knows its language perfectly and is unaffected by such grammatically irrelevant conditions as memory limitations, distractions [...] and errors [...]. Only under the idealization set forth [...] is performance a direct reflection of competence. In actual fact, it obviously could not directly reflect competence.*

*Chomsky (1965), Aspects of the Theory of Syntax, 3f.*

*Theoretical fluid mechanics is an attempt to predict the behavior of real fluid motions by solving boundary value problems [...]. Only the simplest fluid problems can be solved.*
*Therefore, we introduce idealizations into the problems. [...] For example, we could assume that the flow is (a) symmetric, (b) incompressible, (c) not rotating, [...]. The flow, of course, may be none of these, for all are idealizations.*

*Granger (1995), Fluid Mechanics, 17*

**Question:** How and why do falsehoods help us here?

# Idealization and Limits.

Example: The explanation of the rainbow.
Geometrical optics vs. wave optics

Geometrical optics as high-frequency limit of wave optics.
*At* the limit, wave optics breaks down

Wave optics predicts observable deviations from geometrical-optical picture

Dispute (Batterman / Belot): Is the idealization needed?
Belot: No. Wave optics has all the resources
Batterman: Yes. Boundary values from geometrical picture

Batterman (2002), *The Devil in the Details*

Belot (2005) *Philosophy of Science*.

The limit is typically not part of the original class!

# Note of caution: Scientific methodology.



One should not confuse the process of abstraction or idealization with the usual scientific process of isolating stable phenomena from observed data:

James Bogen, James Woodward, Saving the phenomena, *Philosophical Review*: 97(3) (1988):303–352

Benedikt Löwe, Thomas Müller, Data and phenomena in conceptual modelling, *Synthese*, forthcoming

# Back to our example from the beginning (1).

In the spectrum between *Idealization* and *Abstraction*, let us look again at our example from the opening of the talk:

Mathematical concepts and objects have been used throughout the history of mathematics. For each of these concepts/objects, we find a ZFC-representation; in some cases, it is rather natural, in others, less so. This process has certain features of **Idealization**. It is quite different from **Abstraction**, as it chooses one *concrete* formal object to represent the mathematical concept.

Are the ZFC-representations members of the class of concepts they idealize?

- Historically speaking, no!
- In retrospect, yes! (Remember that we said that an accepted formalization has an effect on practice.)

# Back to our example from the beginning (2).



## Are the ZFC-representations members of the class of concepts they idealize?

In our mathematical example, the answer to the question whether the idealization belongs to the original class of representations depends on basic attitudes in philosophy of mathematics: the Platonist will certainly say that the ZFC-representation of $\mathbb{N}$ was around before humans came up with its definition.

# Logical Reasoning (1).

Human rational reasoning was around before logicians tried to codify it. In the Greek tradition logic was the formalization of argumentation for courtroom and politics and argumentation for mathematics.

Modern logicians came up with a formal system intended to model human reasoning which has certain properties, e.g.,

$$p \wedge q \leftrightarrow q \wedge p.$$

But this does not correspond to natural language usage of "and" ("Jeff left the room and Mary cried" vs "Mary cried and Jeff left the room").

Where does this come from; how to account for this given the data? (How does natural reasoning induce commutative conjunction?)

# Logical reasoning (2).

Extensively studied as an empirical subject: Psychology of Reasoning. (Standard example: Wason selection task for material implication.)

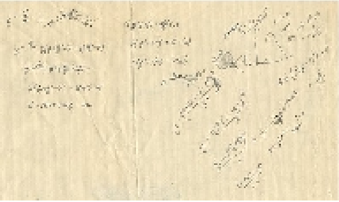
Marian Counihan. Looking for logic in all the wrong places: an investigation of language, literacy and logic in reasoning. PhD Thesis 2008. DS-2008-10.

Keith Stenning, Michiel van Lambalgen (2008). Human reasoning and cognitive science. Cambridge, MA.: MIT Press.

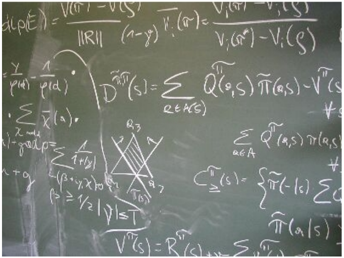
In this example, the identification of the borders between scientific method, idealization, and normative social influence (and preformatting) is extremely difficult.

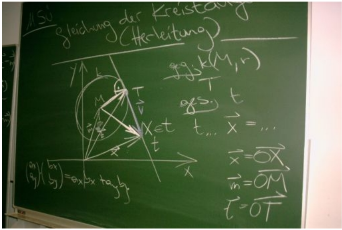
The tradition of anti-psychologism reads the idealization step normatively: everything but the perfect logical reasoner is making mistakes. Does the perfect logical reasoner belong to the original concept?

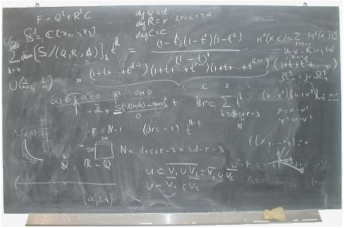
# Mathematical Reasoning (1).

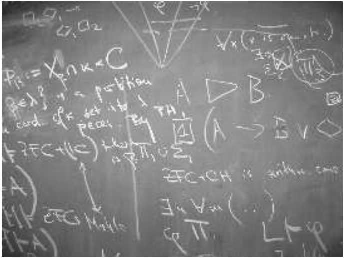


# Mathematical Reasoning (2).







# Mathematical Reasoning (3).

SIMPLE INDUCTIVE MEASURE ANALYSIS 15

$$\begin{array}{rcll} \kappa^{(n+1)} & = & (\kappa^{(n)})^+ = (\kappa^\kappa/\mathcal{C}^\omega_\kappa \otimes n)^+ & \text{(Induction hypothesis)} \\ & \leq & \kappa^\kappa/\mathcal{C}^\omega_\kappa \otimes (n+1) & \text{(Proposition 16)} \\ & \leq & (\kappa^\kappa/\mathcal{C}^\omega_\kappa \otimes n)^\kappa/\mathcal{C}^\omega_\kappa & \text{(Lemma 13)} \\ & = & (\kappa^{(n)})^\kappa/\mathcal{C}^\omega_\kappa \leq \kappa^{(n+1)} & \text{(Induction hypothesis)} \\ & \leq & \kappa^{(n+1)}. & \text{(Lemma 22)} \end{array}$$

We denote the conjunction of $(\dagger_1)$ and $(\dagger_2)$ by $(\mathrm{IH}_*)$.

We proceed by induction on $\xi$, using the following induction hypothesis:[12]

$(\mathrm{IH}_\xi)$ For all $\alpha \leq \xi$, the following three conditions hold:

1. $(\kappa^{(\alpha)})^\kappa/\mathcal{C}^{\omega_1}_\kappa = \kappa^{(\omega+1+\alpha)}$,
2. If $\alpha = \beta + 1$ is a successor and $1 + \beta = \omega \cdot m + n$ then $(\kappa^{(\alpha)})^\kappa/\mathcal{C}^{\omega_1}_\kappa = \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \otimes (m+1) \oplus \mathcal{C}^\omega_\kappa \otimes n = (\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \otimes m \oplus \mathcal{C}^\omega_\kappa \otimes n)^\kappa/\mathcal{C}^{\omega_1}_\kappa$
3. $\mathrm{cf}(\kappa^{(\omega+1+\alpha)}) := \begin{cases} \omega & \text{if } \alpha > 0 \text{ is a limit,} \\ \kappa^+ & \text{if } \alpha \text{ is 1 or a double successor, or} \\ \kappa^{(\omega+1)} & \text{if } \alpha \neq 1 \text{ is zero or a single successor.} \end{cases}$

Obviously, if $(\mathrm{IH}_*)$ and all $(\mathrm{IH}_\xi)$ (for $\xi < \omega^2$) hold, the theorem is proven.

By assumption, $(\kappa^{(0)})^\kappa/\mathcal{C}^{\omega_1}_\kappa = \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa = \kappa^{(\omega+1)}$ and from Theorem 3 (1), we know that this is a regular cardinal, so $(\mathrm{IH}_0)$ holds.

For the successor step $\xi \mapsto \xi + 1$ assume that $(\mathrm{IH}_\xi)$ holds. We first prove parts 1. and 2. of $(\mathrm{IH}_{\xi+1})$. Since $\xi + 1$ is a successor ordinal, $1 + \xi = \omega \cdot m + n$ holds, where $m, n \in \omega$ and not both are zero. We consider the three cases:

*Case 1:* $m = 0$ and $n > 0$, *i.e.*, $\xi + 1 = i + 1$, $i \in \omega$. We have to prove

$$\kappa^{(\omega+1+i+1)} = (\kappa^{(i+1)})^\kappa/\mathcal{C}^{\omega_1}_\kappa = \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \oplus \mathcal{C}^\omega_\kappa \otimes (i+1) = (\kappa^\kappa/\mathcal{C}^\omega_\kappa \otimes (i+1))^\kappa/\mathcal{C}^{\omega_1}_\kappa,$$

which we shall do by induction.

For $i = 0$ we have

$$\begin{array}{rcll} \kappa^{(\omega+1+1)} = (\kappa^{(\omega+1)})^+ & = & (\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa)^+ & \text{(Assumption)} \\ & \leq & \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \oplus \mathcal{C}^\omega_\kappa & \text{(Proposition 16)} \\ & \leq & (\kappa^\kappa/\mathcal{C}^\omega_\kappa)^\kappa/\mathcal{C}^{\omega_1}_\kappa & \text{(Lemma 13)} \\ & = & (\kappa^+)^\kappa/\mathcal{C}^{\omega_1}_\kappa & \text{(Assumption)} \\ & \leq & \kappa^{(\omega+1+1)}. & \text{(Lemma 22)} \end{array}$$

Assume the statement holds for $i$, then

$$\begin{array}{rcll} \kappa^{(\omega+1+(i+1)+1)} & = & (\kappa^{(\omega+1+i+1)})^+ & \\ & = & (\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \oplus \mathcal{C}^\omega_\kappa \otimes (i+1))^+ & \text{(Induction hypothesis)} \\ & \leq & \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \oplus \mathcal{C}^\omega_\kappa \otimes (i+1+1) & \text{(Proposition 16)} \\ & \leq & (\kappa^\kappa/\mathcal{C}^\omega_\kappa \otimes (i+1+1))^\kappa/\mathcal{C}^{\omega_1}_\kappa & \text{(Lemma 13)} \\ & = & (\kappa^{((i+1)+1)})^\kappa/\mathcal{C}^{\omega_1}_\kappa & (\mathrm{IH}_*) \\ & \leq & \kappa^{(\omega+1+(i+1)+1)}. & \text{(Lemma 22)} \end{array}$$

*Case 2:* $m > 0$ and $n = 0$, *i.e.*, $\xi + 1 = \omega \cdot m + 1$. We have to prove

$$\kappa^{(\omega+1+\omega\cdot m+1)} = (\kappa^{(\omega\cdot m+1)})^\kappa/\mathcal{C}^{\omega_1}_\kappa = \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \otimes (m+1) = (\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \otimes (m))^\kappa/\mathcal{C}^{\omega_1}_\kappa,$$

[12] An ordinal $\gamma$ is a double successor is there is some $\delta$ such that $\gamma = \delta + 2$. An ordinal is a single successor if it's a successor but not a double successor; equivalently, if it is the successor of a limit ordinal.

# Mathematical Reasoning (4).



In examples, we have seen instances of the practice of proving. Before the end of the 19th century, there were no formal derivations. In the attempt to codify what a correct mathematical proof is, logicians came up with the notion of a derivation.

Derivations serve important purposes in the foundations of mathematics. Without a formal notion of proof, negative results ("unprovability") would be impossible.

Derivations represent relatively faithfully the informal notion of proof in (certain parts of) foundations of mathematics: It seems that all correct proofs could *in principle* be transformed into derivations.

# Mathematical Reasoning (5).



Proof vs Derivation: in some contexts, it is admissible (even useful) to replace “proof” with “derivation”.

But –as with the ZFC-representation– we can ask ourselves: when we idealize from arbitrary informal proofs to derivations, is the idealization something that was originally included in the concept or did it get inserted *ex post* by the normative power of the formalization?

If we are studying mathematical practice (e.g., epistemology of mathematics), is it acceptable to replace “proof” with “derivation”?

# Practice of mathematical proof (1).



There is very little empirical work on mathematical practice:

> *"[d]ie Soziologie [begegnet] der Mathematik mit einer eigentümlichen Mischung aus Devotion und Desinteresse"*

Bettina Heintz, *Die Innenwelt der Mathematik. Zur Kultur und Praxis einer beweisenden Disziplin*, Wien: Springer, 2000.

- Conference series *Perspectives on Mathematical Practice*, Brussels.
- DFG *Wissenschaftliches Netzwerk* PhiMSAMP: "Philosophy of Mathematics: Sociological Aspects and Mathematical Practice"

# Practice of mathematical proof (2).



Don Fallis, Intentional Gaps in Mathematical Proofs, *Synthese* 134 (2003):45-69

Derivations as a limit case of proofs: each informal proof can be made more precise by filling in steps. The formal derivation is the limit of this process.

But is this an *ex post* metaphor? And is it the real story?

It is not true that a proof on a napkin can be uniquely transformed into a derivation. Making it "more precise" requires decisions of the formalizer, and by formalizing the argument, we are distorting it.

# Practice of mathematical proof (3).



Eva Müller-Hill. Formalizability and knowledge ascriptions in mathematical practice. *Philosophia Scientiae* 13(2). 2009.

Benedikt Löwe, Thomas Müller, Eva Müller-Hill. Mathematical knowledge: a case study in empirical philosophy of mathematics. In: Jonas De Vuyst, Bart Van Kerkhove (*eds.*), Principles of Mathematical Practice 2007. Proceedings. 2009.

Closer look at the analysis of mathematical knowledge in terms of access to derivations: Does the usage of knowledge ascriptions of mathematicians support the important role of derivations demanded by foundationalists in philosophy of mathematics?

# Conclusions



Philosophy of Science has understood that idealizations, even if they result in false statements, are pragmatically legitimate and can be philosophically useful. Idealizations are not taken at face value and there is no normative pressure in philosophy of science to reify them.

In philosophy of logic and mathematics, formalization is a type of idealization, but philosophers are reluctant to admit that the formal representation is a distortion.

Using the normative power of formalization in philosophy of mathematics leads to philosophical analyses that do not correspond to actual mathematical practice inasmuch it differs from the ideal.