Being tolerant about vagueness

Robert van Rooij

Based on joint work with
Pablo Cobreros, Paul Egre, and Dave Ripley

Vagueness and Tolerance

Vagueness: adjectives and orders

- Vagueness with all types of terms

- nouns (‘heap’), verbs (‘sing’), determiners (‘many’)

- But linguistically most work done on *adjectives*

- that give rise to *orders* and *indistinguishability*

- Examples:

Tall	‘Taller than’	indistinguishable tall
Flat	‘Flatter than’	indistinguishable flat

Sorites: Indifference non-transitive

- "A person may be indifferent between 100 and 101 grains of sugar in his coffee, indifferent between 101 and 102, ..., and indifferent between 4999 and 5000. If indifference were transitive he would be indifferent between 100 and 5000 grains, and this is probably false". (Luce, 1956)

- Obviously $Sweet(5000)$ and $\neg Sweet(100)$.
  If $Sweet(5000)$, then $Sweet(4999)$
  if $Sweet(4999)$, then $Sweet(4998)$
  ...
  if $Sweet(101)$, then $Sweet(100)$
  But by repeated modes ponens we derive $Sweet(100)$

Tolerance → not weak order

- A **weak order** is a structure $\langle I, \geq \rangle$, with '$\geq$' a binary relation on I that satisfies the following conditions:
  (R) $\forall x : x \geq x$.
  (TR) $\forall x, y, z : (x \geq y \wedge y \geq z) \rightarrow x \geq z$.
  (Con) $\forall x, y : (x \geq y \vee y \geq x)$.

- $x \sim y$ iff$_{def}$ $x \geq y$ and $y \geq x$. (reflexive, symmetric)

- '$\sim$' is also transitive → equivalence relation.

- Numerical representation (Measurement theory):
  $x \geq y$ iff $f(x) \geq f(y)$ and $x \sim y$ iff $f(x) = f(y)$.

Tolerance $\rightarrow$ Luce's Semi-orders

- A **semi order** is a structure $\langle I, > \rangle$, s.t.
  (IR) $\forall x : x \not> x$.
  (IO) $\forall x, y, v, w : (x > y \wedge v > w) \rightarrow (x > w \vee v > y)$.
  (STr) $\forall x, y, z, v : (x > y \wedge y > z) \rightarrow (x > v \vee v > z)$.

- $x \sim y$ iff$_{def}$ $x \not> y$ and $y \not> x$.
  $\Rightarrow$ reflexive, transitive, but **not transitive**

- Numerical representation (Measurement theory):
  $x > y$ iff $f(x) > f(y) + \epsilon$ and $x \sim y$ iff $f(x) - f(y) < \epsilon$.
  ϵ measures *tolerance*

- Larger tolerance $\epsilon \quad \rightsquigarrow \quad$ coarser grained

Tolerance and Vagueness

- Measurement Tolerance: $\forall x, y : (x \sim_P y \rightarrow (x \not>_P y \wedge y \not>_P x)$

- Kamp (1981), Wright, Dummett:

  What makes a predicate vague, is that it is tolerant

- Tolerance principle: $[\mathbf{P}]\ \forall x, y : (Px \wedge x \sim_P y) \rightarrow Py$

- Tolerance is **constitutive** of its meaning.

Solutions to the Sorites

Solutions to the Sorites

All: give up on [**P**] $\forall x, y \in I : (P(x) \wedge x \sim y) \rightarrow P(y)$

- Supervaluation: [**P**] false,
  but no particular instantiation (super)true.
- Other solutions: *weakening* of [**P**] is true.
  1. Fuzzy logic: [**P**] *almost* true
  2. Three-valued logic
  3. Contextual solution
  4. Pragmatic solution
  5. Epistemics
- None of them takes tolerance seriously (enough)!

Sorites: Three valued logic

- $[\mathbf{P}] \quad \forall x, y \in I : (P(x) \wedge x \sim y) \rightarrow P(y)$

- This gives immediately rise to Sorites paradox.

- Weaken $[\mathbf{P}]$ s.t. (i) still intuitive, but (ii) no paradox

- $[\mathbf{P_{sh}}] \quad \forall x, y \in I : (P(x) \wedge x \sim y) \rightarrow \neg\overline{P}(y)$

- If $[\mathbf{P_{sh}}] \neq [\mathbf{P}]$, then $\neg P \neq \overline{P}$

- Three valued logic, or making use of antonyms.

Sorites: Contextual solution (e.g. Pinkal, Veltman)

- $[\mathbf{P}] \quad \forall x, y \in I : (P(x) \wedge x \sim y) \rightarrow P(y)$

- Assume similarity depends on context (set)
  $x \sim_P^c y$ iff $\neg \exists z \in c : x \sim z > y$ or $x > z \sim y$

- $x \sim_P^c y$ means that x and y are not (even) *indirectly* indistinguishable w.r.t. elements of c

- $(\mathbf{P}_c) \quad \forall x, y, c : (P(x, c) \wedge x \sim_P^c y) \rightarrow P(y, c)$
  is now tautology

- Paradox, confuse with $\forall x, y, c : (P(x, c) \wedge x \sim_P^{\{x,y\}} y) \rightarrow P(y, c)$

Pragmatics: inappropriate contexts

- [**P**] $\forall x, y \in I, c \in C : (P(x,c) \wedge x \sim y \wedge y \in c) \rightarrow P(y,c)$

- Gives rise to Sorites paradox, if we assume that **all** subsets of I are **appropriate** comparison classes.

- Natural conclusion: we should give up this assumption!
$\Rightarrow$ weakening [**P**]

- $c \notin C$ iff '$\sim$' connects all objects in c.

- example: $x \sim y \sim z \sim v \quad \Rightarrow \{x, y, z, v\} \notin C$.

Sorites: Epistemic analysis

- [**P**] $\forall x, y \in I : (P(x) \wedge x \sim y) \rightarrow P(y)$
- This gives immediately rise to Sorites paradox.
- Weaken [**P**] s.t. (i) still intuitive, but (ii) no paradox
- Not just assume that $P(x)$ is true, but that it is **known that** $P(x)$ is true.
- $(\mathbf{P}_w)$ $\forall x, y \in I : (\Box P(x) \wedge x \sim y) \rightarrow P(y)$
- Williamson's epistemic view on vagueness.

Taking tolerance seriously

Tolerant interpretation

Start with FOL model, with $\sim_P$ a non-transitive similarity relation

- $M \models^c \phi$ defined in the usual way.
- $\boxed{M \models^t P(\underline{a})}$ iff $\boxed{\exists d \sim_P a}$: $\boxed{M \models^c P(\underline{d})}$, with $\underline{d}$ name for d
- $M \models^t \neg\phi$ iff $M \not\models^t \phi$
- $M \models^t \phi \wedge \psi$ iff $M \models^t \phi$ and $M \models^t \psi$
- $M \models^t \forall x\phi$ iff for all $d \in I_M : M \models^t \phi[^x/_{\underline{d}}]$.

- $[\![P\underline{a}]\!]^c \subset [\![P\underline{a}]\!]^t$ $\rightsquigarrow$ Tolerant truth **weaker**

- $[\![\neg P\underline{a}]\!]^c \supset [\![\neg P\underline{a}]\!]^t$ $\rightsquigarrow$ Tolerant truth **stronger**

A symmetric reanalysis

- $M \models^t P(\underline{a})$ iff $\exists d \sim_P a : M \models P(\underline{d})$, with $\underline{d}$ as name for d
- $M \models^t \neg\phi$ iff $\boxed{M \not\models^s \phi}$
- $M \models^t \phi \wedge \psi$ iff $M \models^t \phi$ and $M \models^t \psi$
- $M \models^t \forall x\phi$ iff for all $d \in I_M : M \models^t \phi[{}^x/_{\underline{d}}]$.

- $M \models^s P(\underline{a})$ iff $\boxed{\forall d \sim_P a : M \models^c P(\underline{d})}$, with $\underline{d}$ as name for d
- $M \models^s \neg\phi$ iff $\boxed{M \not\models^t \phi}$
- $M \models^s \phi \wedge \psi$ iff $M \models^s \phi$ and $M \models^s \psi$
- $M \models^s \forall x\phi$ iff for all $d \in I_M : M \models^s \phi[{}^x/_{\underline{d}}]$.

Properties

- $[\![\phi]\!]^s \subseteq [\![\phi]\!] \subseteq [\![\phi]\!]^t \quad \rightsquigarrow \quad$ Also for $\neg Pa$

- $Pa \wedge \neg Pa$ not a tolerant contradiction

- $\models^t \quad \approx \quad$ Priest's logic of paradox (kind of relevance logic)
  $\models^{tt}$ Modus Ponens not valid ($\phi, \neg\phi \vee \psi \not\models^{tt} \psi$)

- $Pa \vee \neg Pa$ not strictly valid

- $\models^s \quad \approx \quad$ Kleene's three valued logic

- **Unique**: **combination** of the two!

Solution to Sorites

- $[\mathbf{P}]$ $\quad \forall x, y \in I(P(x) \wedge x \sim y) \rightarrow P(y)$ **is Tolerantly Valid!** or
  $[\mathbf{P'}]$ $\quad \forall x, y \in I(P(x) \wedge x \sim y) \rightarrow^{ct} P(y)$ class valid w.r.t. $\rightarrow^{ct}$
  $[\mathbf{P''}]$ $\quad \forall x, y \in I(P(x) \wedge x \sim y) \rightarrow^{sc} P(y)$ class valid w.r.t. $\rightarrow^{sc}$

- Sorites: $\boxed{\phi \models^{ct} \psi \text{ iff } [\![\phi]\!]^c \subseteq [\![\psi]\!]^t}$ or $\boxed{\phi \models^{sc} \psi \text{ iff } [\![\phi]\!]^s \subseteq [\![\psi]\!]^c}$
  $\boxed{\text{Non transitive}}$ with $[\![\phi]\!]^\alpha = \{M : M \models^\alpha \phi\}$

- If $a \sim_P b \sim_P c$ and $a >_P c$ and $I_M(P) = \{a\}, I_{M'}(P) = \{a, b\}$

- then $Pa \models^{ct} Pb$ (because $M \models^c Pa$ and $M \models^t Pb$)
  and $Pb \models^{ct} Pc$ (because $M' \models^c Pb$ and $M' \models^t Pc$)
  But $Pa \not\models^{ct} Pc$ (because $M \models^c Pa$ but not $M \models^t Pc$)

“Modal” reanalysis: relation with Williamson

- $M \models^c \Box\phi$ iff $M \models^s \phi$ $\quad$ $M \models^c \Diamond\phi$ iff $M \models^t \phi$

- $[\mathbf{P'}] \Rightarrow \forall x, y \in I : (P(x) \wedge x \sim y) \rightarrow \Diamond P(y)$ valid

- $\Box\phi$ is **dual** of $\Diamond\phi$ $\quad$ ($M \models \Diamond\phi$ iff $M \models \neg\Box\neg\phi$)

- $[\mathbf{P''}] \Rightarrow \forall x, y \in I : (\Box P(x) \wedge x \sim y) \rightarrow P(y)$ valid

- Similar to Williamson!
  For him $\Box\phi \not\models \Box\Box\phi$ because Acces-relation non-transitive

- $[\mathbf{P}] \Rightarrow \forall x, y \in I : (\Box P(x) \wedge x \sim y) \rightarrow \Diamond P(y)$ valid

Comparison Super/sub valuationism

- $M \models^{supv} \phi$ iff $\forall M' \geq M : M' \models \phi$

- $M \models^{subv} \phi$ iff $\exists M' \geq M : M' \models \phi$

- for atomic sentences:
  $M \models^{s} Pa \approx M \models^{supv} Pa \qquad M \models^{t} Pa \approx M \models^{subv} Pa$

- Not in general: ours compositional, theirs not

- Entailment with us more traditional
  1. $\phi, \psi \models \phi \wedge \psi$ (not for **Sub**valuationism)
  2. $\phi \vee \psi \models \phi, \psi$ (not for **Super**valuationism)

Pragmatic analysis of Penumbral Connections

- Fine: **Global** quantification crucial for Penumbral connections

- Indeed, our analysis non-standard for $Pa \wedge \neg Pa \quad Pa \vee \neg Pa$

- But we don't see why this is problematic

- But if a slightly shorter than $b \Rightarrow M \models^t Ta \wedge \neg Tb$ is possible!!

- **Pragmatics**: always interpret as *strongly as possible* $\Rightarrow$ strictly

- Strictly interpretation of $Pa \wedge \neg Pb \Rightarrow a >_P b$
  But this is **impossible** if a slightly shorter than b

Conclusions

- Tolerance is constitutive for vagueness.
- New analysis of Sorites: [**P**] tolerantly valid
- Entailment relation non-transitive
- Penumbral connections: Pragmatics
- Pragmatic analysis in accordance with empirical data:
  - Those who accept "$Pa \wedge \neg Pa$" (can only tolerantly true)
  - Accept neither "Pa" nor "$\neg Pa$" (can both be strictly true)
  - Possible if always interpret as strong as possible