Approval as an Intrinsic Part of Preference*

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Abstract

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Approval Voting calls for an extension of the Arrovian preference aggregation model, by incorporating elements of cardinality and interpersonal comparability into individual preferences through assuming the existence of a common zero. We revisit Approval Voting as well as other concepts of Social Choice Theory within this extended model.

1 The Model

The collective decision making problem can be conceived as the aggregation of a vector of utility functions whose informational content depends on the assumptions made about the cardinality and interpersonal comparability of individual preferences. To be more explicit, we consider a non-empty set N of individuals and a non-empty set A of alternatives. Letting $U(A)$ be the set of real-valued "utility functions" defined over A, we model the problem through an aggregation function $f : U(A)^N \rightarrow 2^A \backslash \{\emptyset\}$. The assumptions about the cardinality and interpersonal comparability of individual preferences are formalized by partitioning $U(A)^N$ into information sets, while requiring f to be invariant at any two vector of utility functions which belong to the same information set. At one extreme, one can assume the existence of an absolute scale over which the utilities of individuals are measured and compared. This assumption partitions $U(A)^N$ into singleton information sets, hence imposing no invariance over f. At the other extreme, one can rule out any kind of cardinal information and interpersonal comparability, in which case an information set consists of the elements of $U(A)^N$ which are ordinally equivalent, i.e., induce the same ordering of alternatives for every individual.[1] When cardinality and interpersonal comparability are ruled out, the problem can be modeled through an aggregation function $f : W(A)^N \rightarrow 2^A \backslash \{\emptyset\}$.where $W(A)$ is the set of weak orders (i.e., complete and transitive binary relations) over A. We refer to this as the *Arrovian model* (Arrow (1950, 1951)).

While many voting rules are covered by the Arrovian model[2], *Approval Voting* (AV) falls apart: It generates the social outcome by aggregating vectors of subsets of A. Formally speaking, it is an aggregation function

[1]Given any ordered list $\phi = (\phi_i)_{i \in N}$ of functions from the reals to the reals and any $u \in U(A)^N$, we define $\phi \circ u \in U(A)^N$ as $(\phi \circ u)_i(x) = \phi_i(u_i(x))\ \forall x \in A, \forall i \in N$. When an absolute scale exists, $u, v \in U(A)^N$ are in the same information set iff $v = \phi \circ u$ for some ϕ where each ϕ_i is the identity function. When cardinality and interpersonal comparability are ruled out, $u, v \in U(A)^N$ are in the same information set iff $v = \phi \circ u$ for some ϕ where each ϕ_i is monotonically increasing. As Sen (1986), Bossert and Weymark (2004) eloquently survey, there is a variety of cases between the two extremes.

[2]see, for example, Brams and Fishburn (2002).

$v : (2^A)^N \to 2^A \backslash \{\emptyset\}$ where $S_i \in 2^A$ is conceived as the set of alternatives which are "approved" by $i \in N$. Given any $S \in (2^A)^N$, AV picks the alternatives which are approved by the highest number of individuals. So writing $n(z; S) = \#\{i \in N : z \in S_i\}$ for the number of individuals who approve $z \in A$ at S, we have $v(S) = \{x \in A : n(x; S) \geq n(y; S)\ \forall y \in A\}$.

The literature exhibits various attempts to place AV within the Arrovian model. This is typically done by interpreting AV as a game from μ where 2^A is the common message space of individuals and $v : (2^A)^N \to 2^A \backslash \{\emptyset\}$ is the outcome function. The combination of μ with individuals' preferences over A induces a game whose outcomes are considered. This is a basic mechanism design approach where the approval of an individual is a mere strategic action with no intrinsic meaning. As Dellis (2010), Laslier and Maniquet (2010), Laslier and Sanver (2010b), Nunez (2010) in this volume testify, this interpretation is rich in its variants regarding the modelling and solution of the game. Nevertheless, the same chapters would manifest a dilemma that traps the mechanism design approach: Under natural mechanisms and with mild assumptions over individual preferences, the set of equilibrium outcomes explodes and this set can be refined to the expense of fairly strong assumptions.

We propose to express AV in a framework which partitions $U(A)^N$ into information sets which are finer than those of the Arrovian model. We assume the existence of two cardinal qualifications, "good" and "bad", with a common meaning among individuals. This can be interpreted as the existence of a real number, say 0, whose meaning as a utility measure is common to all individuals. Thus, an information set consists of the ordinally equivalent elements of $U(A)^N$ where 0 is common to all individuals.[3] We call this framework the *extended (Arrovian) model.* In the extended model, the problem can be modeled through an aggregation function $f : W(A \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$ where $W(A \cup \{\emptyset\})$ is the set of weak orders over $A \cup \{\emptyset\}$. Here the empty-set stands for the separation between good and bad: An alternative which is ranked above (resp., below) the empty set is qualified as good (resp., bad). Henceforth, "approval" is not a strategic action but has an intrinsic meaning: It refers to those alternatives which are qualified as good.

Note that every aggregation function expressed in the Arrovian model can also be expressed in the extended one. In fact, aggregation functions of the Arrovian model coincide with those of the extended model which satisfy the following approval independence condition: We say that $f : W(A \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$ is *approval independent* iff $f(R) = f(R')$ for every $R, R' \in W(A \cup \{\emptyset\})^N$ with $x\ R_i\ y \Longleftrightarrow x\ R'_i\ y\ \forall x, y \in A$, $\forall i \in N$.

[3]In other words, $u, v \in U(A)^N$ are in the same information set iff $v = \phi \circ u$ for some $\phi = (\phi_i)_{i \in N}$ where each ϕ_i is monotonically increasing and $\phi_i(0) = 0$.

Although the literature contains studies which imply the extended model,[4] we are not aware of any formal treatment of it. So we explore the extended model, with particular emphasis on Approval Voting. In Section 2, we introduce the *majoritarian approval* axiom which we use as a benchmark of the extended model. In Section 3, we consider four social choice rules, including Approval Voting, under this benchmark. In Section 4, we evaluate these social choice rules according to two criteria, namely monotonicity and independence. In Section 5, we make some concluding remarks, including the possibility of further extending the extended model.

2 A Benchmark: The Majoritarian Approval Axiom

Sertel and Yilmaz (1999) introduce, within the Arrovian model, a "majoritarian approval" axiom which requires from a social choice rule to pick among the alternatives which receive the "approval" of a majority of voters. This requirement explicitly assumes that a voter "approves" an alternative if and only if he ranks it among the first half of his ordering. Such an artificial meaning attributed to the term "approval" is undesirable, but also inevitable within the informational framework of the Arrovian model. On the other hand, thanks to the additional information incorporated by the extended model, majoritarian approval can be naturally redefined. In fact, within the extended model, it is possible to the aggregate the qualifications "good" and "bad" that individuals attribute to alternatives. In other words, based on individual qualifications attributed to an alternative, it is meaningful to speak about that alternative being "socially good" or "socially bad". To express this more formally, let $q(x) \in \{G, B\}^N$ be a *qualification profile* for $x \in A$, where $q_i(x) = G$ (resp., $q_i(x) = B$) means that individual $i \in N$ qualifies x as good (resp., bad). At every $R \in W(A \cup \{\emptyset\})^N$, we write $q(x; R)$ for the qualification profile for x induced by R, i.e., $q_i(x; R) = G \iff x \ P_i \ \emptyset$ holds for all $i \in N$.[5] The aggregation of qualification profiles into a social qualification means to map the set $\{G, B\}^N$ into the set $\{G, B\}$. While this is a separate matter of interest, we will take majoritarianism as granted. Let

[4] Niemi (1984) distinguishes between "approving alternative x" and "voting for alternative x under Approval Voting". Brams and Sanver (2006) mention the possibility of conceiving approval as an intrinsic part of preference. This idea is developed by Brams and Sanver (2009) who, within the general model, propose two new social choice rules. Peters et al. (2009) define Approval Voting as a social choice rule whose domain is preference and approval profiles.

[5] We write P_i for the strict counterpart of R_i.

$n^G(x;R) = \#\{i \in N : q_i(x;R) = G\}$ be the number of individuals who qualify x as good at R. We write $\gamma(R) = \{x \in A : n^G(x;R) \geq \frac{n}{2}\}$ for the (possibly empty) set of alternatives which are qualified as "socially good" at R. We say that $f : W(A \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$ satisfies *majoritarian approval* if and only if we have $g(R) \subseteq \gamma(R)$ at every $R \in W(A \cup \{\emptyset\})^N$ where $\gamma(R) \neq \emptyset$. So, based on how individuals qualify alternatives, majoritarian approval determines the socially good and socially bad ones according to the majority rule, while ruling out the possibility of choosing socially bad alternatives when there are socially good ones.

As we show below, majoritarian approval contradicts approval independence, hence aggregation rules of the Arrovian model all fail majoritarian approval.

Theorem 2.1 *Majoritarian approval and approval independence are logically incompatible.*

Proof. Take any $f : W(A \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$ which is approval independent and satisfies majoritarian approval. Consider a society $N = \{1, 2, 3\}$ and let $R \in W(A \cup \{\emptyset\})^N$ be such that $a\ P_1\ \emptyset\ P_1\ b\ P_1\ c$, $b\ P_2\ a\ P_2\ \emptyset\ P_2\ c$, $c\ P_3\ \emptyset\ P_3\ b\ P_3\ a$. Majoritarian approval implies $g(R) = \{a\}$. Now let $R' \in W(A \cup \{\emptyset\})^N$ be such that $a\ P'_1\ \emptyset\ P'_1\ b\ P'_1\ c$, $b\ P'_2\ \emptyset\ P'_2\ a\ P'_2\ c$, $c\ P'_3\ b\ P'_3\ \emptyset\ P'_3\ a$. Approval independence implies $g(R') = \{a\}$, which contradicts majoritarian approval. ■

3 Four "New" Social Choice Rules

Under the majoritarian approval axiom, the collective decision making problem boils down to answering the following two questions:

(i) How to refine the set of socially good alternatives, when this set contains more than one alternative?

(ii) Which alternative to choose when none of them is socially good?

Throughout the chapter, we will refer to these questions as Question 1 and Question 2.

A common answer to both questions is to pick the alternatives which are qualified as good by the highest number of individuals. This is *Approval Voting* which is formally expressed by the aggregation function $f : W(A \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$ defined as $f(R) = \{x \in A : n^G(x;R) \geq n^G(y;R)\ \forall y \in A\}$ at every $R \in W(A \cup \{\emptyset\})^N$. It is straightforward to check that Approval Voting satisfies majoritarian approval.

Approval Voting, while falling out of the Arrovian model, has a very natural fit to the extended one. In fact, the debate on whether Approval Voting is "indeterminate" or "responsive" now vanishes[6]: The fact is that the informational framework of the Arrovian model is not sufficient to express Approval Voting.[7]

Nevertheless, Approval Voting only uses the information about how individuals qualify the alternatives, hence overlooking the information about rankings. Under Approval Voting, we have $f(R) = f(R')$ for any $R, R' \in W(A \cup \{\emptyset\})^N$ with $q_i(x; R) = q_i(x; R')$ $\forall x \in A, \forall i \in N$. It goes without saying that this overlooked information can be used to define other social choice criteria. We exemplify two of these which satisfy majoritarian approval and which differ in their answers to Question 1 or Question 2.

First, we revisit an aggregation rule of the Arrovian model, namely the Majoritarian Compromise (MC) of Sertel (1986) which, within the Arrovian model, is defined as follows:

(i) The highest-ranked candidate of all voters is considered. If a majority of voters agree on one highest-ranked candidate, this candidate is the MC winner. The procedure stops, and we call this candidate a level 1 winner.

(ii) If there is no level 1 winner, the next-highest ranked candidate of all voters is considered. If a majority of voters agree on one candidate as either their highest or their next-highest ranked candidate, this candidate is the MC winner. If more than one candidate receives a majority support, then the candidate with highest support is the MC winner. The procedure stops, and we call this candidate a level 2 winner.

(iii) If there is no level 2 winner, the voters descend −one level at a time− to lower and lower ranks, stopping when, for the first time, one or more candidates receive a majority support. If exactly one candidate receives a majority support, then this candidate is the MC winner. If more than one candidate receives a majority support, then the candidate with the highest majority support is the MC winner.

We know from Sertel (1986), Sertel and Yilmaz (1999) and from Brams and Kilgour (2001) that the MC winner always arises at a level which does not exceed $\frac{\#A}{2}$. It is worth noting that MC understands "majority" in a weak sense, so as to refer to a coalition whose cardinality is at least as big as the cardinality of its complement.

As one can see in Hoag and Hallett (1926, pp.485-491), MC is the rein-

[6]Laslier and Sanver (2010a) give an account of the exchange between Saari and Newenhizen (1988a, 1988b) and Brams et al. (1988a, 1988b).

[7]This is as if the Borda rule is expressed in a model which aggregates the top ranked alternatives of voters into a social outcome - hence needing the assume the rest of individual rankings. See Endriss et al. (2009) for an analysis of ballot languages.

vention of a voting rule, known as *Bucklin voting*, invented by James W. Bucklin, a lawyer and founder of Grand Junction, Colorado, who proposed his system for Grand Junction in the early 20th century, where it was used from 1909 to 1922 -as well as in other cities- but it is no longer used today. Interestingly, Bucklin asks voters to rank as many of the alternatives they wish, but not necessarily all of them. Given the available rankings of voters, Bucklin voting operates precisely as MC, with the impossibility of descending further in the rankings of certain voters who did not rank all alternatives. Clearly, under Bucklin voting, one can reach the lowest ranked alternative of each voter and still not get a majority, in which case the alternatives with the highest support are elected. Although Bucklin voting is formally absent in the designation of good and bad candidates, those candidates that a voter ranks can be implicitly assumed to be the good ones and that the voter qualifies as bad those he did not care to rank. Thus Bucklin voting can indeed be seen as an adaptation of MC to the extended model, where the descent in a voter's ranking stops when the empty-set is reached. If the descent reaches the empty-set in all voters' rankings and yet no candidate is qualified as socially good, then the alternatives which are qualified good by the highest number of individuals are chosen. We call this adaptation of MC, Majoritarian Approval Compromise (MAC). To illustrate how MAC operates in the extended model, consider the following preference profile with 4 alternatives and 9 voters:

3 voters $a|\ b\ c\ d$
2 voters $b\ a\ c|\ d$
2 voters $c|\ a\ b\ d$
2 voters $d\ b\ c|\ a$

The orderings go from left to right, i.e., the first 3 voters prefer a to b, b to c and c to d, etc. The symbol "|" represents the empty set, i.e., separating the good alternatives from the bad ones. So the first three voters see alternative a as good and the rest as bad, etc. In this profile, initially a gets an approval of 3 while b, c and d get an approval of 2 voters. So none of the alternatives receives a majority approval of 5. We can lower the stick for the b and d voters only (as the a and c voters reached the border between what is good and bad). Now a gets 5 votes, b gets 4 votes, c and d get 2 votes. Hence a is the MAC winner.[8]

MAC satisfies majoritarian approval. Moreover, it coincides with AV

[8]Brams and Sanver (2009) consider the problem of introducing new social choice rules within the general model and what they propose under the name of *Fallback Voting* is what we call MAC in this paper. We also wish to note the similarity between MAC and "fallback bargaining with an impasse" which is a bargaining solution introduced and analyzed by Brams and Doherty (1993) and Brams and Kilgour (2001).

when there are no socially good alternatives. In other words, MAC and AV agree in their answer to Question 2, by picking the alternatives which are qualified as good by the highest number of individuals. On the other hand, they answer Question 1 differently: Among the alternatives which are socially good, AV chooses those which are qualified as good by the highest number of individuals (e.g., alternative c in the above example) while MAC picks those which are qualified as socially good at the earliest level.

Preference-Approval Voting (PAV)[9] is a social choice rule which also differs in its answer to Question 1: It refines the set of socially good outcomes through the construction of the pairwise majority relation among these. When socially good alternatives are multiple, it constructs the pairwise majority relation among the set of socially good alternatives; picks the Condorcet winner if it exists and otherwise, among the alternatives in the top-cycle picks those which are qualified as good by the highest number of individuals. Clearly, PAV satisfies majoritarian approval.

As a final example, we present *Approval Voting with a runoff* (AVR). Given any $R \in W(A \cup \{\emptyset\})^N$, let $\rho(R) = \{x, y\}$ be the pair of alternatives - called *runoff winners*- which receive the highest approval, i.e, $n^G(x; R) \geq n^G(z; R)$ and $n^G(y; R) \geq n^G(z; R)$ hold for any $z \in A \backslash \{x, y\}$.[10] AVR picks the pairwise majority winner among the runoff winners. Remark that AVR is an adaptation of the well-known *plurality with a runoff* defined within the Arrovian model where the runoff winners is the pair of alternatives which are ranked at the top by the highest number of voters. When $\#\gamma(R) > 1$, we have $\rho(R) \subseteq \gamma(R)$, hence the AVR winner is approved by a majority. On the other hand, when $\#\gamma(R) = 1$, AVR may fail to pick the (unique) alternative which is approved by a majority, hence failing majoritarian approval.[11]

We summarize below the behavior of the four social choice rules, as a function of the cardinality of $\gamma(R)$:

[9] Preference-Approval Voting is proposed by Brams and Sanver (2009) and further studied by Erdelyi et al. (2008).

[10] Such a pair need not be unique of course. For expositional simplicity, we assume an exogeneous total order of alternatives which breaks the ties between the alternatives that receive the same number of approvals.

[11] For example, at the preference profile

1 voter $a|\ b$

1 voter $b\ a\ |$

1 voter $|\ b\ a$

with three voters and two alternatives, b is the AVR winner while a is the only alternative which is approved by a majority.

	$\gamma(P) = \emptyset$	$\#\gamma(P) = 1$	$\#\gamma(P) > 1$
AV	most approved alternative in A	$\gamma(P)$	most approved alternative in $\gamma(P)$
MAC	most approved alternative in A	$\gamma(P)$	the alternative which gets "earliest" in $\gamma(P)$
PAV	most approved alternative in A	$\gamma(P)$	most approved alternative in the top-cycle of the pairwise majority relation over $\gamma(P)$
AVR	majority winner in $\rho(P)$	majority winner in $\rho(P)$	majority winner in $\rho(P)$

In the next section, we evaluate the four social choice rules vis-à-vis the satisfaction of two properties, namely monotonicity and independence.

4 Monotonicity and Independence

Among the variety of monotonicity conditions introduced within the Arrovian model, we consider the weakest one which requires that raising an alternative x in individual preference rankings without changing the preference relation on pairs of alternatives that do not include x, cannot have an effect on the election outcome which is detrimental to x. To state this formally, given any $x \in A$ and any $R, R' \in W(A)^N$, we say that *R' is a lifting of x with respect to R* if and only if for every $i \in N$ we have $[x\ R_i\ y \Longrightarrow x\ R'_i\ y\ \forall y \in A]$, $[x\ P_i\ y \Longrightarrow x\ P'_i\ y\ \forall y \in A]$ and $[y\ R_i\ z \Longleftrightarrow y\ R'_i\ z\ \forall y, z \in A\backslash\{x\}]$. A social choice rule $f : W(A)^N \to 2^A\backslash\{\emptyset\}$ is *monotonic* if and only if $x \in f(R) \Longrightarrow x \in f(R')$ whenever R' is a lifting of x with respect to R.[12] We adapt monotonicity to the extended framework as follows: Given any $x \in A$ and any $R, R' \in W(A \cup \{\emptyset\})^N$, we say that *$R'$ is a lifting of x with respect to R* if and only if for every $i \in N$ we have $[x\ R_i\ y \Longrightarrow x\ R'_i\ y\ \forall y \in A]$, $[x\ P_i\ y \Longrightarrow x\ P'_i\ y\ \forall y \in A]$, $[x\ P_i\ \emptyset \Longrightarrow x\ P'_i\ \emptyset]$ and $[y\ P_i\ z \Longleftrightarrow y\ P'_i\ z\ \forall y, z \in (A\backslash\{x\}) \cup \{\emptyset\}]$. A social choice rule $f : W(A \cup \{\emptyset\})^N \to 2^A\backslash\{\emptyset\}$ is *monotonic* if and only if $x \in f(R) \Longrightarrow x \in f(R')$ whenever R' is a lifting of x with respect to R. Note that whenever R' is a lifting of x with respect to R, we have $n^G(x; R') \geq n^G(x; R)$ and $n^G(y; R') = n^G(y; R)\ \forall y \in A\backslash\{x\}$, which implies

[12] This condition, dating back to Fishburn (1982), is originally defined for social choice rules which pick a single alternative at every preference profile. As Sanver and Zwicker (2009) discuss, there is a variety of ways to adapt it to the set-valued context, such as those proposed by Barberà (1977) and Peleg (1979, 1981, 1984).

the following result, whose proof is left to the reader.

Theorem 4.1 *Approval Voting, Majoritarian Approval Compromise, Preference-Approval Voting and Approval Voting with a runoff are all monotonic.*

So monotonicity does not discriminate among the four voting rules we consider. However, in contrast to Approval Voting with a run-off which is monotonic within the extended framework, Plurality with a run-off fails monotonicity within the Arrovian framework (see p.235 of Moulin (1988)).

To define independence, we consider some alternative x^* which is not in A and we write $B = A \cup \{x^*\}$. Writing $W(B \cup \{\emptyset\})$ for the set of weak orders over $B \cup \{\emptyset\}$, from now on, we conceive a social choice rule as a mapping $f : W(A \cup \{\emptyset\})^N \cup W(B \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$ such that $x^* \in f(R)$ only if $R \in W(B \cup \{\emptyset\})^N$. Note that all four voting rules introduced in Section 3 are also defined as a social choice rule $f : W(B \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$. Hence, they are naturally defined as a social choice rule $f : W(A \cup \{\emptyset\})^N \cup W(B \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$.

We say that $R \in W(A \cup \{\emptyset\})$ and $R' \in W(B \cup \{\emptyset\})$ *agree* if and only if for every $i \in N$ and for every $x, y \in A$, we have $x\ R_i\ y \iff x\ R'_i\ y$ and $x\ R_i\ \emptyset \iff x\ R'_i\ \emptyset$. We call x^* a *spoiler* iff $x^* \notin f(R') \neq g(R)$ at some $R \in W(A \cup \{\emptyset\})$ and $R' \in W(B \cup \{\emptyset\})$ which agree. So x^* is called a spoiler if its presence as an alternative can change the social choice without x^* being chosen. A social choice rule $f : W(A \cup \{\emptyset\})^N \cup W(B \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$ satisfies *independence* iff f does not admit any spoiler x^*.[13]

Theorem 4.2 *Approval Voting satisfies independence.*

Proof. Let $f : W(A \cup \{\emptyset\})^N \cup W(B \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$ be Approval Voting. Take any $R \in W(A \cup \{\emptyset\})$ and $R' \in W(B \cup \{\emptyset\})$ which agree. So $\{x \in A : n^G(x; R) \geq n^G(y; R)\ \forall y \in A\} = \{x \in A : n^G(x; R') \geq n^G(y; R')\ \forall y \in A\}$. Thus, if $x^* \notin f(R')$, then $f(R) = f(R')$, establishing the independence of Approval Voting. ■

Theorem 4.3 *Majoritarian Approval Compromise, Preference-Approval Voting and Approval Voting with a runoff fail independence.*

[13] Independence is a well-known choice theoretic property called "Postulate 5*" by Chernoff (1954), "Strong Superset Property" by Bordes (1979), "absorbence" by Sertel and van der Bellen (1979), "Outcast" by Aizerman and Aleskerov (1995). This is also the independence condition which Nash (1950) imposes over a bargaining solution.

Proof. To show that MAC fails independence, consider the following preference profile R with five voters and two alternatives

3 voters $a\ b|$

2 voters $b|\ a$

where a is the unique MAC winner. Now consider the preference profile R' with

2 voters $a\ b|\ x^*$

1 voter $x^*a\ b|$

2 voters $b\ |x^*a$

where b is the unique MAC winner. Moreover R and R' agree, hence MAC fails independence.

To show that PAV fails independence, consider the following preference profile R with three voters and two alternatives:

2 voters $a\ b|$

1 voter $b|\ a$

Both a and b are socially good and a majority of voters prefer a to b, so PAV picks a. Now consider the preference profile R' with

1 voter $a\ b|\ x^*$

1 voter $b\ x^*|a$

1 voter $x^*\ a\ b|$

R and R' agree. All three alternatives are socially good at R' and there is a majority cycle over them, hence PAV picks the one which receives the highest approval which is b, hence failing independence.

To show that AVR fails independence, consider the following preference profile R with nine voters and three alternatives:

4 voters $a|\ b\ c$

3 voters $b|\ a\ c$

2 voters $c|\ b\ a$

where the runoff winners are $\{a, b\}$ among which the pairwise majority winner b is the unique AVR winner. Now consider the preference profile R' with

4 voters $a\ |b\ c\ x^*$

3 voters $x^*b\ |a\ c$

2 voters $c|\ b\ a\ x^*$

where the runoff winners are $\{a, x^*\}$ among which the pairwise majority winner a is the unique AVR winner. As R and R' agree, AVR fails independence. ∎

Nevertheless, MAC, PAV and AVR can be evaluated according to the "popularity" of the spoiler they admit. After all, social choice rules that admit spoilers with little public support are more open to manipulation via artificial candidacies than those where the spoiler must have a reasonably high public support. We show that while MAC performs very poor in this regard, under PAV and AVR, a spoiler must have a reasonably high public support.[14]

Theorem 4.4 (i) *Under Majoritarian Approval Compromise, for any number of voters, there may be a spoiler who is approved by only one voter.*

(ii) *Under Preference-Approval Voting, x^* is a spoiler only if x^* is socially qualified as good.*

(iii) *Under Approval Voting with runoff, x^* is a spoiler only if x^* is a runoff winner.*

Proof. To show (i), consider a preference profile $R \in W(A \cup \{\emptyset\})$ where the society is split in two coalitions K and $N\backslash K$ whose rankings are as follows:

Voters in K: $a\ b|$...............

Voters in $N\backslash K$: $b|\ a$ ....

There are at least two alternatives called a and b. Voters in K qualify a and b as good; voters in $N\backslash K$ qualify b as good. It does not matter whether there are other alternatives and if so, how they are ranked. We also let the cardinality of K and $N\backslash K$ differ by at most one while $\#K \geq \#N\backslash K$. So at R, if $\#K = \#N\backslash K$, then $\{a, b\}$ is the MAC winner and if $\#K > \#N\backslash K$, then $\{a\}$ is the MAC winner. Now take a voter $i \in K$ and consider the preference profile and $R' \in W(B \cup \{\emptyset\})$ with

Voters in $K\backslash\{i\}$: $a\ b|\ x^*$.........

Voter i: $x^*a\ b|$........

Voters in $N\backslash K$: $b\ |x^*a$............

At R', if $\#K = \#N\backslash K$, then $\{b\}$ is the MAC winner and if $\#K > \#N\backslash K$, then $\{a, b\}$ is the MAC winner. Note that R and R' agree while x^* is not chosen at R'. Hence x^* is a spoiler. Moreover, x^* is approved by only one voter.

To show (ii), let $f : W(A\cup\{\emptyset\})^N \cup W(B\cup\{\emptyset\})^N \to 2^A\backslash\{\emptyset\}$ be PAV. Take any $R \in W(A\cup\{\emptyset\})$ and $R' \in W(B\cup\{\emptyset\})$ which agree while $x^* \notin f(R')$. Note that if $\#\gamma(R) \in \{0, 1\}$, then $f(R) = f(R')$, hence x^* is not a spoiler. Now

[14] This is in contrast to Plurality with runoff which, as defined in the Arrovian model, is hurt by the existence of spoilers with very low public support. More precisely, in the Arrovian model, independent of the number of voters, it is possible to construct an example where the spoiler is the best alternative for two voters and the worst alternative for the rest of the voters. So when monotonicity and independence are the salient criteria to evaluate social choice rules, Approval Voting with runoff presents a neat improvement over Plurality with runoff. This justifies a comment in a similar direction made by Rida Laraki at the workshop on "Reforming the French Presidential Electoral System: Experiments on Electoral Reform", held at CEVIPOF, Sciences-Po, Paris, on 15-16 June 2009.

let $\#\gamma(R) \geq 2$. If $x^* \notin \gamma(R')$, then $\gamma(R) = \gamma(R')$, implying $f(R) = f(R')$, hence x^* is not a spoiler.

To show (iii), let $f : W(A \cup \{\emptyset\})^N \cup W(B \cup \{\emptyset\})^N \to 2^A \backslash \{\emptyset\}$ be AVR. Take any $R \in W(A \cup \{\emptyset\})$ and $R' \in W(B \cup \{\emptyset\})$ which agree. Suppose $x^* \notin \rho(R')$. As R and R' agree, we have $\rho(R) = \rho(R')$ and also $f(R) = f(R')$, hence x^* is not a spoiler. ■

5 Concluding Remarks

Approval Voting calls for an extension of the Arrovian model by incorporating elements of cardinality and interpersonal comparability into individual preferences, through assuming the existence of a common zero. This naturally occurs in certain environments, such as matching models (see, for example, Roth and Sotomayor (1990)), where "being self-matched" is the common zero. However, in general, a common zero is implied by the existence of a common meaning attributed to "good" and "bad". This is a minimal divergence from the Arrovian model whose information sets are refined by the use of monotonic transformations which have one fixed point.[15]

The Arrovian model can be further extended through the use of monotonic transformations having multiple fixed points.[16] This incorporates further elements of cardinality and interpersonal comparability. In fact, at the extreme case of requiring every point to be fixed, the identity function becomes the only allowed monotonic transformation, hence getting back to the existence of an absolute scale to measure utilities.

Extending the Arrovian model invites interesting philosophical questions some of which are discussed by Ng (1992). Moreover, as the degree of incorporated cardinality and interpersonal comparability can be mesaured by the number of fixed points imposed over the monotonic transformations, the extent to which, if any, the Arrovian model can be extended invites interesting experimental questions as well. In any case, we see these extensions as interesting conceptual tools which, as this section suggests, allow to revisit and better understand certain concepts of social choice theory.

[15] Instead of using any monotonic transformation. See Footnotes 1 and 3.

[16] In fact, the literature contains studies and proposals of social choice rules that call for further extensions. (see, for example, Hillinger (2005), Aleskerov et al. (2007), Balinski and Laraki (2007)).

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