Dialogues, End-Rules and Definitional Reasoning

Thomas Piecha

(joint work with Peter Schroeder-Heister)

University of Tübingen
Wilhelm-Schickard-Institute

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Outline

Language

(i) propositional *formulas* $A, B, \ldots$
atomic formulas $a, b, \ldots$
logical constants $\neg, \wedge, \vee, \rightarrow$

(ii) *special symbols* $\vee$, $\wedge_1$ and $\wedge_2$

(iii) letters P ('proponent') and O ('opponent')

(iv) an *expression* e is either a formula or a special symbol. For each expression there is one *P-signed expression* $P\,$e and one *O-signed expression* $O\,$e.

(v) a signed formula is called *assertion*;
a signed special symbol is called *symbolic attack*.

A dialogue for $a \rightarrow (b \wedge a)$

positions

0.	P	$a \rightarrow (b \wedge a)$	
1.	O	a	$[0, A]$
2.	P	$b \wedge a$	$[1, D]$
3.	O	$\wedge_2$	$[2, A]$
4.	P	a	$[3, D]$

moves

Argumentation Forms

X and Y, where $X \neq Y$, are variables for P and O.

negation $\neg$:	assertion:	$X \neg A$	
	attack:	$Y A$	
	defense:	*no defense*	
conjunction $\wedge$:	assertion:	$X A_1 \wedge A_2$	
	attack:	$Y \wedge_i$	(Y chooses $i = 1$ or $i = 2$)
	defense:	$X A_i$	
disjunction $\vee$:	assertion:	$X A_1 \vee A_2$	
	attack:	$Y \vee$	
	defense:	$X A_i$	(X chooses $i = 1$ or $i = 2$)
implication $\rightarrow$:	assertion:	$X A \rightarrow B$	
	attack:	$Y A$	
	defense:	$X B$	

Example for implication

0.	P	$a \rightarrow (b \rightarrow a)$	
1.	O	a	$[0, A]$
2.	P	$b \rightarrow a$	$[1, D]$

Dialogues

Dialogue (1)

A *dialogue* is a sequence of moves

(i) made alternatingly by P and O

(ii) according to the argumentation forms,

(iii) and P makes the first move.

Dialogue (2)

$(D10)$ P may assert an atomic formula only if it has been asserted by O before.

$(D11)$ If at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p.

$(D12)$ An attack may be defended at most once.

$(D13)$ A P-signed formula may be attacked at most once.

A dialogue beginning with $P\,A$ is called *dialogue for the formula A*.

Proponent P and opponent O are not interchangeable due to $(D10)$ and $(D13)$.

Dialogues

P wins a dialogue

P wins a dialogue for a formula A if

(i) the dialogue is finite,

(ii) begins with the move $P\,A$ and

(iii) ends with a move of P such that O cannot make another move.

Example, dialogue won by P

0.	P	$(a \vee b) \to \neg\neg(a \vee b)$	
1.	O	$a \vee b$	$[0, A]$
2.	P	$\vee$	$[1, A]$
3.	O	a	$[2, D]$
4.	P	$\neg\neg(a \vee b)$	$[1, D]$
5.	O	$\neg(a \vee b)$	$[4, A]$
6.	P	$a \vee b$	$[5, A]$
7.	O	$\vee$	$[6, A]$
8.	P	a	$[7, D]$

Dialogue not won by P

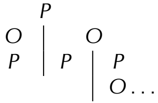
0.	P	$(a \vee b) \to \neg\neg(a \vee b)$	
1.	O	$a \vee b$	$[0, A]$
2.	P	$\neg\neg(a \vee b)$	$[1, D]$
3.	O	$\neg(a \vee b)$	$[2, A]$
4.	P	$a \vee b$	$[3, A]$

Strategies

A dialogue for formula *A* won by *P* is **not** a proof of *A*!

Dialogue tree

A *dialogue tree* contains all possible dialogues for *A* as paths.



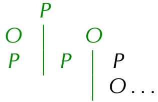
Strategy

A *strategy* for a formula *A* is a subtree *S* of the dialogue tree for *A* such that

(i) *S* does not branch at even positions (i.e. at *P*-moves),

(ii) *S* has as many nodes at odd positions as there are possible moves for *O*,

(iii) all branches of *S* are dialogues for *A* won by *P*.



A strategy for *A* is a proof of *A*.

Strategies

Example, strategy for $(a \vee b) \to \neg\neg(a \vee b)$

0.			P $(a \vee b) \to \neg\neg(a \vee b)$			
1.			O $a \vee b$			$[0, A]$
2.			P $\vee$			$[1, A]$
3.	O	a	$[2, D]$	O	b	$[2, D]$
4.	P	$\neg\neg(a \vee b)$	$[1, D]$	P	$\neg\neg(a \vee b)$	$[1, D]$
5.	O	$\neg(a \vee b)$	$[4, A]$	O	$\neg(a \vee b)$	$[4, A]$
6.	P	$a \vee b$	$[5, A]$	P	$a \vee b$	$[5, A]$
7.	O	$\vee$	$[6, A]$	O	$\vee$	$[6, A]$
8.	P	a	$[7, D]$	P	b	$[7, D]$

(There are other strategies.)

Theorem (Felscher 1985)

There is a strategy for a formula A iff A is provable in intuitionistic logic.

PROOF by showing for Gentzen's sequent calculus *LJ* (for intuitionistic logic) that every strategy can be transformed into a proof in *LJ*, and vice versa.

Definitions

Definitional clause

A *definitional clause* is an expression of the form

$$a \Leftarrow B_1 \wedge \ldots \wedge B_n$$

for $n \geqslant 0$, where a is atomic and B_i can be complex.

Definition

A finite set $\mathcal{D}$ of definitional clauses

$$\mathcal{D} \begin{cases} a \Leftarrow \Gamma_1 \\ \quad \vdots \\ a \Leftarrow \Gamma_k \end{cases}$$

is a *definition* of a, where $\Gamma_i = B^i_1 \wedge \ldots \wedge B^i_{n_i}$ is the body of the i-th clause.

Definitional Closure and Reflection

For a given definition

$$\mathcal{D} \begin{cases} a \Leftarrow \Gamma_1 \\ \vdots \\ a \Leftarrow \Gamma_k \end{cases}$$

we have for sequents:

Principle of definitional closure $(\vdash \mathcal{D})$

$$\frac{\Delta \vdash \Gamma_i}{\Delta \vdash a} \; (\vdash \mathcal{D})$$

Principle of definitional reflection $(\mathcal{D} \vdash)$

$$(\mathcal{D} \vdash) \; \frac{\Delta, \Gamma_1 \vdash C \quad \ldots \quad \Delta, \Gamma_k \vdash C}{\Delta, a \vdash C}$$

(for propositional atoms; for first-order a proviso is needed)

Definitional Closure and Reflection

In sequent calculus:

Proof theory is extended to atomic formulas.

Proofs do not have to begin with atomic formulas.

For dialogues:

Replace end-rule for atomic formulas by end-rule for complex formulas.

Have to allow to continue reasoning for atomic formulas.

But: proof for e.g. $a \to a$ should end with atomic formula a, even if definitional clauses for a are given.

C-Dialogues

C-dialogue

A *C-dialogue* is a dialogue with the additional condition (end-rule)

($D14$) O can attack a formula C if and only if
(i) C has not yet been asserted by O, or
(ii) C has already been attacked by P.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for dialogues are directly carried over to the corresponding notions for C-dialogues.

Difference between C-dialogue and dialogue:

(i) C-dialogue won by P ends with assertion of a complex or atomic formula.

(ii) Dialogue won by P can only end with assertion of an atomic formula.

C-Dialogues

Example, C-strategy for $(a \vee b) \to \neg\neg(a \vee b)$

0.	P	$(a \vee b) \to \neg\neg(a \vee b)$	
1.	O	$a \vee b$	$[0, A]$
2.	P	$\neg\neg(a \vee b)$	$[1, D]$
3.	O	$\neg(a \vee b)$	$[2, A]$
4.	P	$a \vee b$	$[3, A]$

O cannot attack $a \vee b$ since the conditions of $(D14)$ are not satisfied:

(i) $a \vee b$ has already been asserted by O and

(ii) $a \vee b$ has not been attacked by P.

The C-dialogue is won by P, and it is a C-strategy for $(a \vee b) \to \neg\neg(a \vee b)$.

C-Dialogues

Complex initial sequent

$$(\mathrm{Id})\ \frac{}{A \vdash A}\ (A \text{ atomic or complex})$$

Theorem (Isomorphism)

C-strategies and sequent calculus derivations with complex initial sequents are isomorphic.

Important in definitional reasoning where meaning of atomic formulas can be given by complex formulas (corresponds to complex assumptions).

Structural Reasoning

Contraction

Twofold attack by P corresponds to (Contr) in sequent calculus derivations.

Example, $\neg(a \wedge \neg a)$ not provable without twofold attack on $a \wedge \neg a$ by P

0.	P	$\neg(a \wedge \neg a)$	
1.	O	$a \wedge \neg a$	$[0, A]$
2.	P	$\wedge_1$	$[1, A]$
3.	O	a	$[2, D]$
4.	P	$\wedge_2$	$[1, A]$
5.	O	$\neg a$	$[4, D]$
6.	P	a	$[5, A]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a}\,(\text{Id})}{a, \neg a \vdash}\,(\neg\vdash)}{a, a \wedge \neg a \vdash}\,(\wedge\vdash)}{a \wedge \neg a, a \wedge \neg a \vdash}\,(\wedge\vdash)}{a \wedge \neg a \vdash}\,(\text{Contr})}{\vdash \neg(a \wedge \neg a)}\,(\vdash\neg)$$

Contraction-free dialogues

Contraction-free dialogues are dialogues where $(D13)$ ('a P-signed formula may be attacked at most once') is replaced by

$(D13')$ A formula may be attacked at most once.

Definitional Reasoning

Argumentation form

For each atom a defined by $a \Leftarrow B^i_1 \wedge \ldots \wedge B^i_{n_i}$ $(1 \leqslant i \leqslant k)$ *definitional reasoning* determines how $X\,a$ can be attacked by Y and how this attack can be defended by X. 'Γ' is a special symbol indicating the attack.

definitional reasoning:	assertion:	$X\,a$	
	attack:	$Y\,\Gamma$	
	defense:	$X\,\Gamma_i$	(X chooses $i = 1, \ldots, k$)

Correspondence to definitional reflection and closure (sequent calculus)

Definitional reasoning with attack $O\,\Gamma$ corresponds to *definitional closure* $(\vdash \mathfrak{D})$.

Def. reasoning with attack $P\,\Gamma$ corresponds to *definitional reflection* $(\mathfrak{D} \vdash)$.

Definitional Reasoning

Definitional dialogues

Definitional dialogues are C-dialogues

(i) which can start with the assertion of an atomic formula,

(ii) and where atomic formulas can be attacked.

Condition $(D10)$ is replaced by $(D10')$:

$(D10')$ P may assert an atomic formula if it has been asserted by O before, or if it is asserted in a defense to an attack $O\,\Gamma$.

P can now assert atomic formulas in defenses to opponent attacks $O\,\Gamma$ without O having asserted them before.

Definitional Dialogues with Contraction

We consider the (paradoxical) definitional clause $a \Leftarrow \neg a$.
Neither strategy for a nor for $\neg a$. The dialogue trees have only infinite branches.

Comparison to sequent calculus

However, a as well as $\neg a$ are provable for $a \Leftarrow \neg a$ in sequent calculus with definitional reflection $(\mathcal{D}\vdash)$, definitional closure $(\vdash\mathcal{D})$ and contraction (Contr):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a}\,(\mathrm{Id})}{a, \neg a \vdash}\,(\neg\vdash)}{a, a \vdash}\,(\mathcal{D}\vdash)\; a \Leftarrow \neg a}{a \vdash}\,(\mathrm{Contr})}{\dfrac{\vdash \neg a}{\vdash a}\,(\vdash\mathcal{D})}\,(\vdash\neg)
$$

Isomorphism between strategies of definitional dialogues and sequent calculus derivations using $(\mathcal{D}\vdash)$, $(\vdash\mathcal{D})$ and (Contr) needs further restrictions.

Definitional Dialogues with Contraction

Further restrictions

The following condition is added:

$(D16)$ O may attack an atom a by definitional reasoning only if it has not been asserted by O before.

And condition $(D14)$ has to be restricted to nonatomic formulas:

$(D14')$ O can attack a nonatomic formula C if and only if
(i) C has not yet been asserted by O, or
(ii) C has already been attacked by P.

Example, let a be defined by $a \Leftarrow b$

0.	P	$a \rightarrow a$	
1.	O	a	$[0, A]$
2.	P	a	$[1, D]$

is a strategy nevertheless, due to $(D16)$.

Definitional Dialogues with Contraction

Example, $a \Leftarrow \neg a$

For $a \Leftarrow \neg a$ there is a strategy for a as well as for $\neg a$ if condition $(D16)$ is respected:

0.	P	a	
1.	O	Γ	$[0, A]$
2.	P	$\neg a$	$[1, D]$
3.	O	a	$[2, A]$
4.	P	Γ	$[3, A]$
5.	O	$\neg a$	$[4, D]$
6.	P	a	$[5, A]$

0.	P	$\neg a$	
1.	O	a	$[0, A]$
2.	P	Γ	$[1, A]$
3.	O	$\neg a$	$[2, D]$
4.	P	a	$[3, A]$

Definitional Dialogues without Contraction

Contraction-free definitional dialogues

Contraction-free definitional dialogues are def. dialogues with $(D13')$ ('a formula may be attacked at most once'), and where the following condition is added:

$(D17)$ P must not assert an atom that has been attacked by P before.

Example, $a \Leftarrow \neg a$

For $a \Leftarrow \neg a$ the contraction-free dialogue trees for a and $\neg a$ are

0.	P	a			0.	P	$\neg a$	
1.	O	Γ	$[0, A]$		1.	O	a	$[0, A]$
2.	P	$\neg a$	$[1, D]$	resp.	2.	P	Γ	$[1, A]$
3.	O	a	$[2, A]$		3.	O	$\neg a$	$[2, D]$
4.	P	Γ	$[3, A]$					
5.	O	$\neg a$	$[4, D]$					

There is neither a strategy for a nor for $\neg a$.

Definitional Dialogues and Kreuger's Rule

Another possibility:
No restrictions on contraction, but initial sequents are restricted by

Kreuger's rule: $(\mathrm{Id})\,\dfrac{}{a \vdash a}$ iff $a \Leftarrow a$ only clause for a in given definition $\mathcal{D}$.

Then neither $\vdash a$ nor $\vdash \neg a$ are derivable for $a \Leftarrow \neg a$ in $\mathcal{D}$.

Equivalent restriction can be formulated for dialogues.

Then (definitional) reasoning must continue if atom is defined.

Example, $a \Leftarrow \neg a$

For $a \Leftarrow \neg a$ there is neither a strategy for a nor for $\neg a$.

The dialogue trees for a and for $\neg a$ contain only infinite branches.

Definitional Dialogues and Selective Contraction

Problems

Kreuger's rule (with contraction): too restrictive! $\rightsquigarrow$ only total definitions

Without any contraction: too restrictive! $\rightsquigarrow$ no strategy for $\neg(a \wedge \neg a)$

With contraction:
Arbitrary assumptions can be contracted with assumptions given by definition.

0.	P	a	
1.	O	Γ	$[0, A]$
2.	P	$\neg a$	$[1, D]$
3.	O	a	$[2, A]$
4.	P	Γ	$[3, A]$
5.	O	$\neg a$	$[4, D]$
6.	P	a	$[5, A]$

0.	P	$\neg a$	
1.	O	a	$[0, A]$
2.	P	Γ	$[1, A]$
3.	O	$\neg a$	$[2, D]$
4.	P	a	$[3, A]$

Work in progress

How to restrict contraction such that only assumptions *of the same status* can be contracted?

Conclusions

(i) Dialogues were extended to C-dialogues with end-rule for complex formulas.

(ii) C-strategies are isomorphic to sequent calculus derivations with complex initial sequents.

(iii) Isomorphism important to handle complex assumptions.

(iv) C-dialogues were extended to definitional dialogues.

(v) Definitional dialogues enable reasoning about definitions (including paradoxical ones).

(vi) Structural reasoning (contraction) plays critical role in reasoning about paradoxes.