Algebraic Independence-Friendly Logic

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Syntax

- Imagine a first-order sentence:

$$\forall x \exists u \forall y \exists v\, \phi(x, y, u, v)$$

- Branching quantifiers (Henkin 1961):

$$\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} \phi(x, y, u, v)$$

- Independence-friendly logic (Hintikka and Sandu 1989):

$$\forall x \exists u \forall y \exists v_{/x}\, \phi(x, y, u, v)$$

- IFG logic (Dechesne 2005):

$$\forall v_{0/\emptyset} \exists v_{1/\emptyset} \forall v_{2/\{0,1\}} \exists v_{3/\{0,1\}}\, \phi(v_0, v_1, v_2, v_3)$$

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Semantics

- $\forall v_0 \exists v_1 (v_0 \leq v_1)$ — True in $\mathbb{R}$
- $\forall v_{0/\emptyset} \exists v_{1/\{0\}} (v_1 \leq v_0)$ — Neither True nor False in $\mathbb{R}$; True in $\mathbb{N}$
- $\mathbb{R} \models_{\langle 3,7 \rangle} (v_0 \leq v_1)$
  $\mathbb{R} \not\models_{\langle 5,4 \rangle} (v_0 \leq v_1)$
- $\mathbb{R} \models^{+}_{\{\langle 1,2 \rangle, \langle 3,7 \rangle, \langle e,\pi \rangle\}} (v_0 \leq v_1)$
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Semantics

Definition

For any $V \subseteq {}^{N}A$,

- $\mathfrak{A} \models^{+}_{V} \phi$ if Eloïse has a winning strategy for $G(\mathfrak{A}, \phi, V)$,
- $\mathfrak{A} \models^{-}_{V} \phi$ if Abélard has a winning strategy for $G(\mathfrak{A}, \phi, V)$.

Semantics

Proposition (Downward Monotonicity)

If $V' \subseteq V$*, then* $\mathfrak{A} \models^{\pm}_{V} \phi$ *implies* $\mathfrak{A} \models^{\pm}_{V'} \phi$*.*

Proposition (Noncontradiction)

$\mathfrak{A} \models^{+}_{V} \phi$ *and* $\mathfrak{A} \models^{-}_{V} \phi$ *if and only if* $V = \emptyset$*.*

These two properties characterize the meanings of IFG-formulas in finite structures.

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Meanings of formulas

Definition (First-order logic)

$$\phi^{\mathfrak{A}} = \{\, \vec{a} \in {}^{N}\!A \mid \mathfrak{A} \models_{\vec{a}} \phi \,\}$$

Definition (IFG logic)

$$\|\phi\|^{\pm}_{\mathfrak{A}} = \{\, V \subseteq {}^{N}\!A \mid \mathfrak{A} \models^{\pm}_{V} \phi \,\}$$

$$\|\phi\|_{\mathfrak{A}} = \langle \|\phi\|^{+}_{\mathfrak{A}}, \|\phi\|^{-}_{\mathfrak{A}} \rangle$$

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$\mathfrak{Cs}_{\mathsf{IFG}_N}(\mathfrak{A})$

Definition

$$\mathfrak{Cs}_{\mathsf{IFG}_N}(\mathfrak{A}) = \{\|\phi\| : \phi \in \mathscr{L}^{\sigma}_{IFG_N}\}$$

$$1 = \langle \mathscr{P}({}^{N}A), \{\emptyset\}\rangle$$

$$0 = \langle \{\emptyset\}, \mathscr{P}({}^{N}A)\rangle$$

$$D_{ij} = \|v_i = v_j\|$$

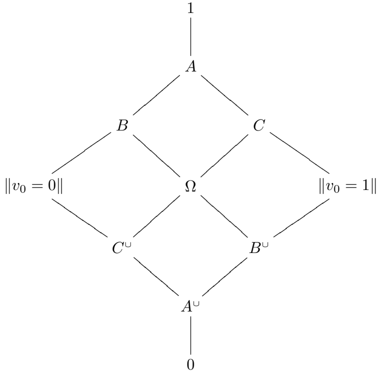
$$\|\phi\|^{\cup} = \|{\sim}\phi\|$$

$$\|\phi\| +_J \|\psi\| = \|\phi \vee_{/J} \psi\|$$

$$\|\phi\| \cdot_J \|\psi\| = \|\phi \wedge_{/J} \psi\|$$

$$C_{n,J}(\|\phi\|) = \|\exists v_{n/J}\phi\|$$

$\mathfrak{Cs}_{\mathsf{IFG}_1}(2)$



- $1 = \langle \mathscr{P}\{0,1\}, \{\emptyset\} \rangle$
- $A = \langle \mathscr{P}\{0\} \cup \mathscr{P}\{1\}, \{\emptyset\} \rangle$
- $B = \langle \mathscr{P}\{0\}, \{\emptyset\} \rangle$
- $C = \langle \mathscr{P}\{1\}, \{\emptyset\} \rangle$
- $\|v_0 = 0\| = \langle \mathscr{P}\{0\}, \mathscr{P}\{1\} \rangle$
- $\|v_0 = 1\| = \langle \mathscr{P}\{1\}, \mathscr{P}\{0\} \rangle$
- $\Omega = \langle \{\emptyset\}, \{\emptyset\} \rangle$
- $\ldots$
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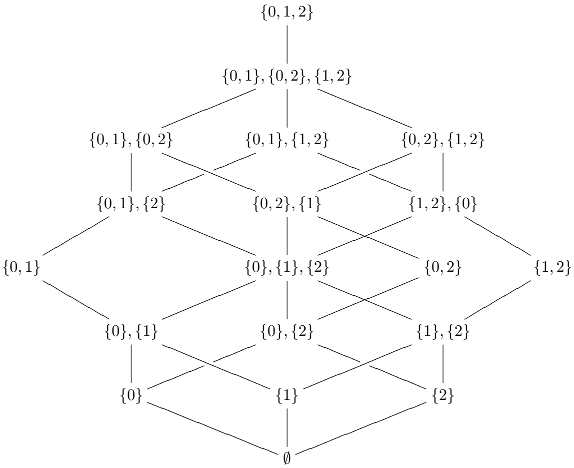
$\mathfrak{Cs}_{\mathsf{IFG}_1}(3)$



Perfect IFG-formulas

Definition

The *perfection* of an IFG_N-formula ϕ, denoted $\phi_\emptyset$, is the formula obtained by emptying all its independence sets, e.g.,

$$\begin{aligned} \phi &\quad \text{is} \quad \forall v_{1/J}(v_0 = v_1 \vee_{/K} v_0 \neq v_1), \\ \phi_\emptyset &\quad \text{is} \quad \forall v_{1/\emptyset}(v_0 = v_1 \vee_{/\emptyset} v_0 \neq v_1). \end{aligned}$$

Note that $\phi_\emptyset$ is equivalent to a first-order formula.

Proposition

- $\|\phi\|^+ \subseteq \|\phi_\emptyset\|^+$ *and* $\|\phi\|^- \subseteq \|\phi_\emptyset\|^-$.
- $\|\phi_\emptyset\| = \langle \mathscr{P}(V), \mathscr{P}({}^N A \setminus V)\rangle$.
- ϕ *is equivalent to a first-order formula iff* $\|\phi\| = \|\phi_\emptyset\|$.

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Principle of Trivalence

Proposition

Let ϕ be a sentence, and let $\emptyset \neq V \subseteq {}^N A$.

$$\text{If } \mathfrak{A} \models^{\pm}_{V} \phi, \text{ then } \mathfrak{A} \models^{\pm}_{^N A} \phi.$$

Corollary

If ϕ is a sentence, then $\|\phi\| \in \{0, \Omega, 1\}$.

- $1 = \langle \mathscr{P}({}^N A), \{\emptyset\} \rangle$
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De Morgan Algebras

Definition

A *De Morgan algebra* is a bounded distributive lattice with a negation that satisfies

$$\sim\sim x = x \quad \text{and} \quad \sim(x \vee y) = \sim x \wedge \sim y.$$

Definition

A *Kleene algebra* is a De Morgan algebra that also satisfies

$$x \wedge \sim x \leq y \vee \sim y.$$

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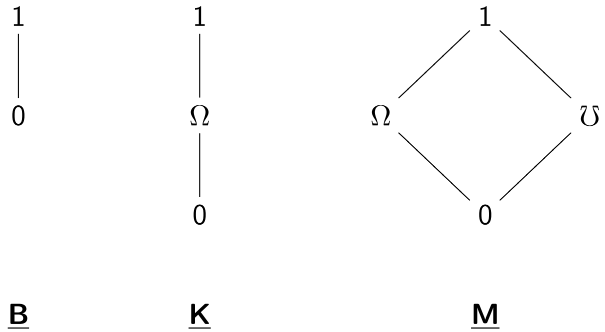
Boolean ⊂ Kleene ⊂ De Morgan



Theorem

The class of reducts of $\mathfrak{Cs}_{\mathrm{IFG}_N}(\mathfrak{A})$ *to the signature*

$$\langle 1, 0, {}^{\cup}, +_N, \cdot_N \rangle$$

generates the variety of Kleene algebras.

Therefore, the correct propositional logic to use with IFG logic is Kleene's strong three-valued logic.

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Monadic De Morgan Algebras

Definition

A *quantifier* on a De Morgan algebra is a unary operation ∇:

- $\nabla 0 = 0$,
- $x \leq \nabla x$,
- $\nabla(x \vee y) = \nabla x \vee \nabla y$,
- $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$.
- $\nabla({\sim}\nabla x) = {\sim}\nabla x$.

A De Morgan algebra equipped with a quantifier is called a *monadic De Morgan algebra*.

Theorem

The class of reducts of $\mathfrak{Cs}_{\mathrm{IFG}_1}(\mathfrak{A})$ *to the signature*

$$\langle 1, 0, ^{\cup}, +_{\{0\}}, \cdot_{\{0\}}, C_{0,\{0\}} \rangle$$

generates a subvariety of the monadic Kleene algebras that satisfy

$$\nabla(x \wedge \sim x) \leq \sim\nabla(x \wedge \sim x).$$

If and only if

- In ordinary first-order logic,

$$\mathfrak{A} \models \phi \leftrightarrow \psi \quad \text{iff} \quad \phi^{\mathfrak{A}} = \psi^{\mathfrak{A}}.$$

- In IFG logic, is there a schema such that

$$\mathfrak{A} \models^{+} \xi(\phi, \psi) \quad \text{iff} \quad \|\phi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}}?$$

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Schema $\longrightarrow$ Term

Definition

Every IFG-schema ξ has a corresponding term T_ξ in the language of IFG-algebras. T_ξ is defined recursively as follows:

- $T_{\alpha_i} = X_i$,
- $T_{v_i = v_j} = D_{ij}$,
- $T_{\sim\xi} = (T_\xi)^\cup$,
- $T_{\xi_1 \vee_{/J} \xi_2} = T_{\xi_1} +_J T_{\xi_2}$,
- $T_{\exists v_{n/J}\xi} = C_{n,J}(T_\xi)$.

"Iff" is not expressible in IFG logic

Proposition

Any $\mathfrak{Cs}_{\mathsf{IFG}_N}(\mathfrak{A})$ *that has a term operation* $T(X, Y)$ *such that*

$$T(X, Y) = 1 \quad \textit{iff} \quad X = Y$$

is hereditarily simple.

Proposition

- $\mathfrak{Cs}_{\mathsf{IFG}_1}(2)$ *is hereditarily simple.*
- $\mathfrak{Cs}_{\mathsf{IFG}_1}(3)$ *is simple, but not hereditarily simple.*

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Theorem

There is no IFG-schema such that

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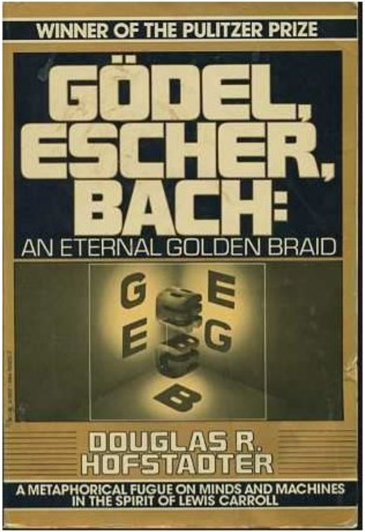
WINNER OF THE PULITZER PRIZE
GÖDEL, ESCHER, BACH:
AN ETERNAL GOLDEN BRAID
DOUGLAS R. HOFSTADTER
A METAPHORICAL FUGUE ON MINDS AND MACHINES IN THE SPIRIT OF LEWIS CARROLL