Definability in Dependence Logic

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Dependence Logic

The syntax of Dependence Logic ($\mathcal{D}$) extends the syntax of FO by new atomic (dependence) formulas of the form

$$=(t_1, \ldots, t_n),$$

with the meaning that the values of the terms $t_1, \ldots, t_{n-1}$ determine the value of t_n.
The semantics of formulas of dependence logic is defined in terms of *teams* which are set of assignments:

Definition

Let A be a set and $\{x_1, \ldots, x_k\}$ set of variables. A *team* X of A with domain $\{x_1, \ldots, x_k\}$ is a set of assignments s from $\{x_1, \ldots, x_k\}$ into A.

Semantics of $\mathcal{D}$

We restrict attention to formulas in negation normal form. The following two operations on teams will be needed:

Definition

Suppose A is a set, X is a team of A, and $F\colon X \to A$.

- Then $X(F/x_n)$ denotes the *supplement* team $\{s(F(s)/x_n) \ : s \in X\}$.
- The *duplicate* team $X(A/x_n)$ is defined as $X(A/x_n) = \{s(a/x_n) \ : s \in X \text{ and } a \in A\}$.

Definition

Let $\mathfrak{A}$ be a model and X a team of A. The satisfaction relation $\mathfrak{A} \models_X \varphi$ is defined as follows:

- $\mathfrak{A} \models_X t_1 = t_2$ iff for all $s \in X$ we have $t_1^{\mathfrak{A}}\langle s\rangle = t_2^{\mathfrak{A}}\langle s\rangle$.
- $\mathfrak{A} \models_X \neg t_1 = t_2$ iff for all $s \in X$ we have $t_1^{\mathfrak{A}}\langle s\rangle \neq t_2^{\mathfrak{A}}\langle s\rangle$.

Semantics continued

- $\mathfrak{A} \models_X =(t_1, ..., t_n)$ iff for all $s, s' \in X$ such that $t_1^{\mathfrak{A}}\langle s\rangle = t_1^{\mathfrak{A}}\langle s'\rangle, \ldots, t_{n-1}^{\mathfrak{A}}\langle s\rangle = t_{n-1}^{\mathfrak{A}}\langle s'\rangle$, we have $t_n^{\mathfrak{A}}\langle s\rangle = t_n^{\mathfrak{A}}\langle s'\rangle$.
- $\mathfrak{A} \models_X \neg =(t_1, ..., t_n)$ iff $X = \emptyset$.
- $\mathfrak{A} \models_X R(t_1, \ldots, t_n)$ iff for all $s \in X$ we have $(t_1^{\mathfrak{A}}\langle s\rangle, \ldots, t_n^{\mathfrak{A}}\langle s\rangle) \in R^{\mathfrak{A}}$.
- $\mathfrak{A} \models_X \neg R(t_1, \ldots, t_n)$ iff for all $s \in X$ we have $(t_1^{\mathfrak{A}}\langle s\rangle, \ldots, t_n^{\mathfrak{A}}\langle s\rangle) \notin R^{\mathfrak{A}}$.
- $\mathfrak{A} \models_X \psi \wedge \phi$ iff $\mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_X \phi$.
- $\mathfrak{A} \models_X \psi \vee \phi$ iff $X = Y \cup Z$ such that $\mathfrak{A} \models_Y \psi$ and $\mathfrak{A} \models_Z \phi$.
- $\mathfrak{A} \models_X \exists x_n \psi$ iff $\mathfrak{A} \models_{X(F/x_n)} \models \psi$ for some $F\colon X \to A$.
- $\mathfrak{A} \models_X \forall x_n \psi$ iff $\mathfrak{A} \models_{X(A/x_n)} \psi$.

Finally, a sentence φ is true in a model $\mathfrak{A}$ if $\mathfrak{A} \models_{\{\emptyset\}} \varphi$.

Goal of the talk

Theorem

For every sentence ϕ of $\mathcal{D}$ there is a sentence ψ of Σ_1^1 s.t.

$$\text{For all } \mathfrak{A}\text{: } \mathfrak{A} \models_{\{\emptyset\}} \phi \iff \mathfrak{A} \models \psi. \quad (1)$$

Conversely, for every sentence ψ of Σ_1^1 there is ϕ of $\mathcal{D}$ s.t. (1) holds.

Our goal is to characterize *definable sets of teams*, i.e., sets of the form

$$\{X : \mathfrak{A} \models_X \phi\}.$$

We know that such sets are always closed downwards:

Theorem (Downward closure)

Suppose $\varphi \in \mathcal{D}$ and $Y \subseteq X$. Then $\mathfrak{A} \models_X \varphi$ implies $\mathfrak{A} \models_Y \varphi$.

Examples of definable properties of teams

Definition

Let A be a set and X a team with domain $\{x_1, \ldots, x_k\}$. Denote by $\mathrm{rel}(X)$ the k-ary relation of A corresponding to X

$$\mathrm{rel}(X) = \{(s(x_1), \ldots, s(x_k)) : s \in X\}.$$

Example

Let A be a set and F a family of sets of n-tuples of A which is closed under subsets. Suppose that there is a $n+1$-ary relation R on A s.t. for every $T \subseteq A^n$,

$$T \in F \Leftrightarrow \text{ there is } b \in A \text{ s.t. } R(\overline{a}b) \text{ for all } \overline{a} \in T.$$

Then it holds that

$$(A, R) \models_X \exists y(=\!(y) \wedge R(\overline{x}, y)) \Leftrightarrow \mathrm{rel}(X) \in F.$$

Example

Let $k \in \mathbb{N}$ and let $P(x)$ be a polynomial with positive integer coefficients. Then there is $\varphi(\overline{x}) \in \mathcal{D}$ s.t. for all finite sets A and teams X over $\{x_1, \ldots, x_k\}$

$$A \models_X \varphi \Leftrightarrow |X| \leq P(|A|).$$

Towards the characterization

We restrict attention first to the special case where $L = \emptyset$, i.e., we look at collections $\{X : A \models_X \phi\}$ where our model is just a pure set.

Definition

Let $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ and R a k-ary predicate. Denote by Q_φ the following class of $\{R\}$-structures

$$Q_\varphi = \{(A, \mathrm{rel}(X)) \mid A \models_X \varphi\}.$$

Lemma

For every $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$*, the class* Q_φ *is closed under isomorphisms.*

Proposition

For every $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ *the class* Q_φ *is the class of models of some sentence in* $\Sigma_1^1[\{R\}]$*.*

Corollary

Let $k \in \mathbb{N}$*. There is no formula* $\varphi(x_1, \ldots, x_k) \in \mathcal{D}[\emptyset]$ *such that for all* A *and teams* X *with domain* $\{x_1, \ldots, x_k\}$*:*

$$A \models_X \varphi \Leftrightarrow |X| \text{ is finite.}$$

On downwards monotonicity

Definition

Let R be k-ary and $\varphi \in \Sigma^1_1[\{R\}]$ a sentence. We say that φ is *downwards monotone with respect to* R if for all A and $B' \subseteq B \subseteq A^n$

$$(A, B) \models \varphi \Rightarrow (A, B') \models \varphi.$$

Proposition

A sentence $\varphi \in \Sigma^1_1[\{R\}]$ *is downwards monotone with respect to* R *iff there is* $\psi \in \Sigma^1_1[\{R\}]$ *such that*

$$\models \varphi \leftrightarrow \psi,$$

and R *appears only negatively in* ψ*.*

Proof of Proposition

Suppose $\varphi \in \Sigma^1_1[\{R\}]$ is downwards monotone with respect to R. Let φ^* be acquired by replacing all the occurrences of R in φ by a new predicate R'. By the downwards monotonicity of φ

$$\models \varphi \leftrightarrow \exists R'(\varphi^* \wedge \forall \overline{x}(R(\overline{x}) \rightarrow R'(\overline{x}))).$$

For the other direction, we use induction on the construction of φ.

The characterization

Lemma (Skolem normal-form)

Every Σ^1_1 *formula is equivalent to a formula of the form*

$$\exists f_1 \ldots \exists f_n \forall x_1 \ldots \forall x_m \psi,$$

where ψ *is a quantifier-free formula.*

Theorem

Let $k \geq 1$ *and* R *a* k*-ary predicate. Suppose that* Q *is a downwards monotone class of* $\{R\}$ *structures. Then there is a formula* $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ *such that* $Q = Q_\varphi$ *if and only if* Q *is* $\Sigma^1_1[\{R\}]$*-definable.*

A sketch of the proof

Assume that Q is downwards monotone and $\Sigma^1_1[\{R\}]$-definable. We will construct a formula $\chi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ such that $Q = Q_\chi$. We may assume that there is $\lambda \in \Sigma^1_1[\{R\}]$

$$\lambda = \exists f_1 \ldots \exists f_n \forall x_1 \ldots \forall x_m \psi$$

defining Q where:

- ψ is in conjunctive normal form,
- for each f_i $(1 \leq i \leq n)$ there are unique pairwise distinct variables $z^i_1, \ldots, z^i_s$, such that all occurrences of f_i in ψ are of the form $f_i(z^i_1, \ldots, z^i_s)$,
- R has in total only one occurrence (say $\neg R(x_1, \ldots, x_k)$) in ψ and it is negative.

Proof continued

We are now ready to define χ now as

$$\forall x_1 \cdots \forall x_m \exists y_1 \cdots \exists y_n (=(\overline{z}_i, y_1) \wedge \cdots \wedge =(\overline{z}_n, y_n) \wedge \psi^+),$$

where ψ^+ is acquired from ψ by:

- replacing all occurrences of $f_i(\overline{z}_i)$ by the variable y_i,
- $\neg R(x_1, \ldots, x_k)$ is replaced by the formula

$$\bigvee_{1 \le i \le k} y_i \neq x_i.$$

The case $L \neq \emptyset$

Theorem

Let L be a vocabulary, $\mathfrak{A}$ a L-model and F a family of sets of k-tuples of A which is closed under subsets. Then the following are equivalent:

1. $F = \{\mathrm{rel}(X) : \mathfrak{A} \models_X \psi(y_1, \ldots, y_k)\}$ *for some* $\psi(y_1, \ldots, y_k) \in \mathcal{D}[L]$.
2. $F = \{Y : (\mathfrak{A}, Y) \models \phi(R)\}$ *for some sentence* $\phi \in \Sigma^1_1[L \cup \{R\}]$, *in which R occurs only negatively.*

Transferring the results to IF-logic

Definition

Let $\varphi(y_1, \ldots, y_k) \in \mathrm{IF}[\emptyset]$ and R a k-ary predicate. Denote by Q_φ the class of $\{R\}$-structures $(A, \mathrm{rel}(X))$ such that X is a *trump* with domain $\{y_1, \ldots, y_k\}$ for $\varphi(y_1, \ldots, y_k)$ in A.

Theorem

Let Q be a downwards monotone class of $\{R\}$-models. Then there is a formula $\varphi(y_1, \ldots, y_k) \in \mathrm{IF}[\emptyset]$ such that $Q = Q_\varphi$ iff Q is $\Sigma^1_1[\{R\}]$-definable.

The case $L \neq \emptyset$

Theorem

Let L be a vocabulary, $\mathfrak{A}$ a L-model and F a family of sets of k-tuples of A which is closed under subsets. Then the following are equivalent:

1. $F = \{\mathrm{rel}(X) : X \textit{ is a trump for } \psi(y_1, \ldots, y_k) \textit{ in } \mathfrak{A}\}$ *for some formula* $\psi(y_1, \ldots, y_k) \in \mathrm{IF}[L]$.
2. $F = \{Y : (\mathfrak{A}, Y) \models \phi(R)\}$ *for some sentence* $\phi \in \Sigma^1_1[L \cup \{R\}]$*, in which R occurs only negatively.*

Application

Consider the following versions $\exists^1$ and $\forall^1$ of the quantifiers of dependence logic: $\exists^1$ is defined by the clause

$$\mathfrak{A} \models_X \exists^1 x_n \psi \text{ iff there is } a \in A \text{ s.t. } \mathfrak{A} \models_{X(a/x_n)} \models \psi,$$

and $\forall^1$ by

$$\mathfrak{A} \models_X \forall^1 x_n \psi \text{ iff for all } a \in A \text{ it holds that } \mathfrak{A} \models_{X(a/x_n)} \models \psi.$$

Note that $\exists^1 x \psi$ can be expressed uniformly as $\exists x (=(x) \wedge \psi)$.

What about $\forall^1$?

Denote by $(\mathcal{D} + \forall^1)$ the extension of $\mathcal{D}$ by $\forall^1$. It is easy to see that (with respect to sentences)

$$(\mathcal{D} + \forall^1) \equiv \Sigma^1_1 \equiv \mathcal{D}\,.$$

Since $(\mathcal{D} + \forall^1)$ remains downwards monotone, our result implies that

$$(\mathcal{D} + \forall^1) \equiv \mathcal{D},$$

with respect to open formulas also.

Question

Is $\forall^1$ "uniformly" definable in $\mathcal{D}$?

Team Logic

Recall that *Team logic* (TL) is acquired by closing $\mathcal{D}$ under classical negation $(\sim)$. Note that with $\sim$, e.g., classical disjunction and the following form of universal quantification: "for all $F\colon X \to A$" can be expressed.

Theorem (Ville Nurmi (2008))

$\mathrm{TL} \equiv \mathrm{SO}$.

Definability in Team Logic

Note that with formulas downward monotonicity does not hold anymore. In fact we can show the following:

Theorem

Let Q be a class of $\{R\}$-structures. Then there is a formula $\varphi(y_1, \ldots, y_k) \in \mathrm{TL}[\emptyset]$ such that $Q = Q_\varphi$ if and only if Q is SO-definable.

Theorem

Let L be a vocabulary, $\mathfrak{A}$ a L-model and F a family of sets of k-tuples of A which is closed under subsets. Then the following are equivalent:

1. $F = \{\mathrm{rel}(X) : \mathfrak{A} \models_X \psi(y_1, \ldots, y_k)\}$ *for some formula* $\psi(y_1, \ldots, y_k) \in \mathrm{TL}[L]$.
2. $F = \{Y : (\mathfrak{A}, Y) \models \phi(R)\}$ *for some sentence* $\phi \in \mathrm{SO}[L \cup \{R\}]$.

References

- Juha Kontinen and Jouko Väänänen: *On Definability in Dependence Logic*. To appear in JoLLI.
- Juha Kontinen and Ville Nurmi: *Team Logic and Second Order Logic*. In preparation.