Logic Constants
Invariance for Modal and Dynamic Logic

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The Gothenburg project

Invariance and constancy:
Logical Foundations for Interaction

PI: D. Westerståhl

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The initial question :

- What is a logical constant?
  Why do we select $\wedge$, $\exists$, $\forall$ as logical expressions and build our logical systems around them?

The good old answer (Tarski's)

- Because they have special semantic properties
  Quantifiers get interpreted by operations which are invariant under permutation.

The Gothenburg project (cont.)

Widen the horizon:

- consider other languages
- consider other objects
- consider alternative approaches to logicality

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  What are the natural 'logical' operations on games?
  What's the connection with linear connectives?
- consider alternative approaches to logicality
  From consequence relations to logical constants.
  Logicality as constancy

For today

Some sample work:
invariance for modal and dynamic logic

- generalizes previous work on FO languages
- through a general perspective
  on invariance and logical systems

J. van Benthem & D. Bonnay, *Modal Logic and Invariance*, JANCL, 2008.

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Languages and similarity relations

L a logic, S a ‘similarity relation’ for L
(equivalence relation on the class of L-structures)

1. L’s expressive power is bound by S,

2. In these limits, L is as expressive as possible:

3. S-invariance as L’s logicality criterion.

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   $\Longrightarrow$ Isomorphisms, Potential isomorphisms for FOL
   $\Longrightarrow$ Bisimulations for Modal Logic
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   - L is the strongest ‘finitary’ logic such that (1) holds.

   $\Longrightarrow$ Lindström Theorem
   $\Longrightarrow$ van Benthem characterization Theorem
3. S-invariance as L’s logicality criterion.

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3. S-invariance as L's logicality criterion.
   - S-closed classes interpret logical operations.

   $\Longrightarrow$ FO quantifiers and invariance under isomorphisms

A case in point

$L = FOL$, $S = Iso_p$ (short for 'being potentially isomorphic')

Definition

f is a **partial isomorphism** between $\mathcal{A}$ and $\mathcal{B}$
just in case f is an isomorphism btw substructures of $\mathcal{A}$ and $\mathcal{B}$.

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A **potential isomorphism** I between $\mathcal{A}$ and $\mathcal{B}$
is a nonempty set of partial isomorphisms s.t.
for every $f \in I$ and $a \in A$ (resp. $b \in B$),
there is $g \in I$ with $f \subseteq g$ and $a \in dom(g)$ (resp. $b \in rng(g)$).

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Classical example: $\langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{R}, \leq \rangle$
not isomorphic but potentially isomorphic.

A case in point (cont.)

- $L = FOL$, $S = Iso_p$

An obvious but elusive parallel:

L	=	atoms, booleans	+	$\exists$
S	=	partial isomorphisms	+	picking one more

A case in point (cont.)

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An obvious but elusive parallel:

L	=	atoms, booleans	+	$\exists$
S	=	partial isomorphisms	+	picking one more

- $L = ML$, $S = BiS$ (short for 'being bisimilar')

Same thing:

L	=	atoms, booleans	+	$\Diamond$
S	=	world matching	+	moving along R

Atom preservation

$$\langle M, P, a\rangle \overset{S}{=\!=\!=} \langle M', P', a'\rangle$$

$$a \in P \text{ iff } a' \in P'$$

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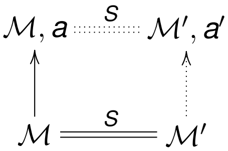
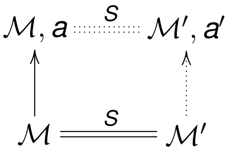
Definition

A similarity relation S **preserves atoms** iff
for all S-similar structures, distinguished relations
behave similarly on distinguished objects.

Commutation with object expansions



Commutation with object expansions



Definition

A similarity relation S **commutes with object expansions** iff
if $\mathcal{M}\ S\ \mathcal{M}'$,
then for all $a \in |\mathcal{M}|$, there is an $a' \in |\mathcal{M}'|$ s.t. $\mathcal{M}, a\ S\ \mathcal{M}', a'$.

Ordering on similarity relations

Definition

$S \leq S'$ iff $S' \subseteq S$.

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Ex : Universal relation $\leq Iso_p \leq Iso$.

Characterization of Iso_p

Fact

Iso_p is the smallest similarity relation S such that

- *S preserves atoms*
- *S commutes with objects expansions.*

Back to the logic

How does this connect with properties of first-order languages ?

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How does this connect with properties of first-order languages ?

Atoms preservation says that similar structures are elementary equivalent on atomic sentences and boolean compounds thereof.

What does commutation says ?

It says that existential quantification is in the language.

Object projection

Definition

Let Q be a class of structures of the form $\mathcal{M}, a$.
The **object projection** of Q, $\exists(Q)$, is defined by $\mathcal{M} \in \exists(Q)$ iff there is a $b \in |\mathcal{M}|$ such that $\mathcal{M}, b \in Q$.

This is what you can do with $\exists$.

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This is what you can do with $\exists$.

$\exists$ is logical *means* $\exists$ does not break invariance:

Definition

Object projection **preserves S-invariance** iff whenever Q is S-invariant, so is $\exists(Q)$.

Equivalence result

Theorem

S commutes with object expansions
iff
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Corollary

Iso_p is the smallest similarity relation S such that

- *S preserves atoms*
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Equivalence result

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Corollary

Iso_p is the smallest similarity relation S such that

- *S preserves atoms*
- *object projection preserves S-invariance.*

$\Longrightarrow$ Iso_p is the good match for a language based on $\exists$.

A general setting

- A a class of objects
- E a relation on A
- S an equivalence relation on A
- $E^{-1} : \wp(A) \rightarrow \wp(A)$ an inverse for E
  defined for $X \subseteq A$ by $E^{-1} = \{a \in A \ / \ \exists b \in X \text{ with } aEb\}$

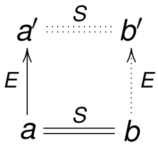
Definition

A subclass X is **S-invariant** iff
if $a \in X$ and aSb then $b \in X$.

Commutation

Definition

S **commutes with** E iff,
for all $a, a', b \in A$, if aSb and aEa',
then there is a b' such that $a'Sb'$ and bEb'.



Preservation of Invariance

Definition

E^{-1} **preserves S-invariance** iff for any subclass X of A, if X is S-invariant, then $E^{-1}(X)$ is S-invariant.

$$\begin{array}{ccc} X & & S\text{-invariant} \\ \downarrow & & \vdots \\ E^{-1}(X) & & S\text{-invariant} \end{array}$$

Commutation lemma

Lemma

S commutes with E iff E^{-1} preserves S-invariance.

Commutation lemma

Proof

S commutes with E, ? E^{-1} preserves S-invariance ?

$\Rightarrow$

$$a \in E^{-1}(X) \xlongequal{S} b$$

Commutation lemma

Proof

S commutes with E, ? E^{-1} preserves S-invariance ?

$\Rightarrow$

$$\begin{array}{lll} a' \in X & & \\ \uparrow E & & \\ a \in E^{-1}(X) & \overset{S}{=\!=\!=} & b \end{array}$$

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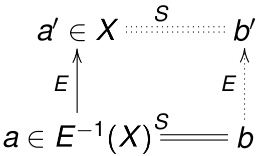

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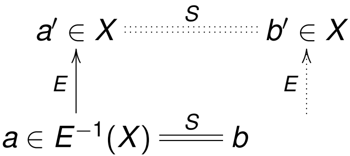

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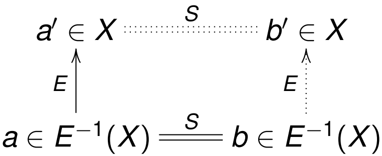
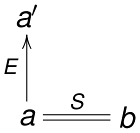


Commutation lemma

Proof

? S commutes with E ?, E^{-1} *preserves S-invariance*

$\Leftarrow$



Commutation lemma

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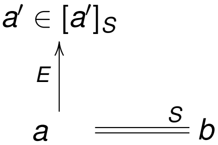

Commutation lemma

Proof

? S commutes with E ?, E^{-1} preserves S-invariance

$\Leftarrow$

$$\begin{array}{ccc} a' \in [a']_S & & \\ \uparrow{\scriptstyle E} & & \\ a \in E^{-1}([a']_S) & \overset{S}{=\!=\!=} & b \end{array}$$

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$$a' \in [a']_S$$

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$$a \in E^{-1}([a']_S) \overset{S}{=\!=\!=} b \in E^{-1}([a']_S)$$

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E \uparrow & & \vdots\uparrow E \\
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\end{array}
$$

Using the lemma

Take

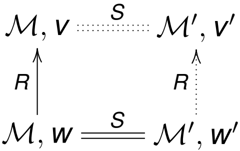
- A the class of FO structures
- E expanding with one object
  $\Rightarrow E^{-1}$ is object projection.

As an instance of the commutation lemma, we get:

Theorem

S commutes with object expansions
iff
object projection preserves S-invariance.

The modal case



Definition

A similarity relation S
commutes with guarded object expansion iff,
if $\mathcal{M}, w \; S \; \mathcal{M}', w'$, then for all $v \in |\mathcal{M}|$ with wRv,
there is a $v' \in |\mathcal{M}'|$ such that $\mathcal{M}, v \; S \; \mathcal{M}', v'$ and $w' R' v'$.

Characterization of bisimulations

BiS short for 'being bisimilar'

Fact

BiS is the smallest similarity relation S such that

- *S preserves atoms*
- *S commutes with guarded object expansion.*

Guarded object projection

Definition

Let Q be a class of pointed Kripke structures.
The **guarded object projection** of Q, $\Diamond(Q)$, is defined by
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This is what you can do with $\Diamond$.

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$\Diamond$ is logical *means* $\Diamond$ does not break invariance:

Definition

Guarded object projection **preserves** ***S*-invariance** iff whenever Q is S-invariant, so is $\Diamond(Q)$.

Equivalence result

- A the class of pointed Kripke structures
- E moving to an accessible world
  $\Rightarrow E^{-1}$ is guarded object projection.

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As an instance of the commutation lemma, we get:

Theorem

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Corollary

Bis is the smallest similarity relation S such that

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Dynamic logic

The language of Propositional Dynamic Logic (PDL)

Programs

$$\pi \;:=\; R \mid \pi;\pi \mid \pi \cup \pi \mid \pi^* \mid ?\phi$$

Formulas

$$\phi \;:= p \mid \neg\phi \mid \phi \wedge \phi \mid \langle\pi\rangle\phi$$

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- ML formulas can only define set of worlds
- PDL programs can also define relations

What is a PDL operation?

Look at what $\cup$ does on a fixed set of worlds W:

$$\begin{aligned} ||\cup||_W \quad : \quad & \wp(W^2) \times \wp(W^2) \rightarrow \wp(W^2) \\ & R \times R' \mapsto R \bigcup R \end{aligned}$$

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In general: a dynamic operator $\overline{O}$ is interpreted by a function O from Kripke models to relations over these models.

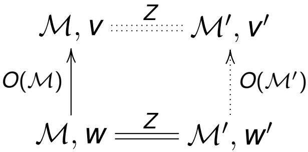
So let $\overrightarrow{\chi}$ be a sequence of programs and formulas matching the syntactic type of $\overline{O}$,

The semantic clause for $\overline{O}$ is given by:

$$||\overline{O}\overrightarrow{\chi}||_{\mathcal{M}} = O(|\mathcal{M}|, ||\overrightarrow{\chi}||_{\mathcal{M}})$$

Safety

For any bisimulation Z:



Definition

A dynamic operation O is **safe for bisimulation** iff whenever Z is a bisimulation between $\mathcal{M}, w$ and $\mathcal{M}', w'$, and $wO(\mathcal{M})v$ for some $v \in |\mathcal{M}|$, then there is a $v' \in |\mathcal{M}'|$ such that vZv' and $w'O(\mathcal{M}')v'$.

Safety and PDL

- Safety is the key element in the proof that PDL formulas are invariant under bisimulation.
- Enriching PDL programs with new safe operations yields extensions which are still invariant under bisimulation.
- PDL without Kleene star is the safe fragment of FOL.

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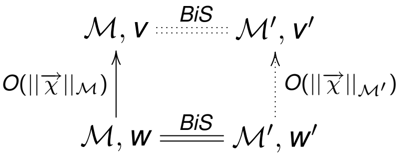
However, safety is not our standard commutation property.

Commutation with *Bis*



Definition

A dynamic operation O **commutes with BiS** iff
whenever $\mathcal{M} + ||\overrightarrow{\chi}||_{\mathcal{M}}, w$ and $\mathcal{M}' + ||\overrightarrow{\chi}||_{\mathcal{M}'}, w'$, and
$wO(||\overrightarrow{\chi}||_{\mathcal{M}})v$ for some $v \in |\mathcal{M}|$,
then there is a $v' \in |\mathcal{M}'|$ such that vZv' and $w'O(||\overrightarrow{\chi}||_{\mathcal{M}'})v'$.

The lemma again

As before, we can get:

Theorem

BiS commutes with O iff O preserves invariance under BiS.

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As before, we can get:

Theorem

BiS commutes with O iff O preserves invariance under BiS.

Preserving invariance under *BiS* is precisely what we need if we want to stay within the realm of modal logic.

Safety and commutation

How does this relate to safety?

Safety and commutation

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Theorem

BiS commutes with O iff O is safe for bisimulation.

Safety is indeed the natural constraint on dynamic operations *qua* modal.

Conclusion

- *a general perspective on invariance,*
  'commutation lemma'
  easy but nice result for Iso_p and *BiS*
  nice and not so easy result for safety
- *stemming from some sort or 'reverse' meta-logic,*
  Duality btw syntax and semantics
  Take *S* as a parameter
- *to be developed...*
  Apply to other logical systems
  Generalize to games via game logic
  Connect to 'intrinsic' regularities