Inclusion and exclusion atoms in team semantics

Pietro Galliani

Institute for Logic, Language and Computation
Universiteit van Amsterdam

Finite Model Theory Seminar

Outline

Outline

Independence Logic

Independence Atoms (Grädel, Väänänen)

$M \models_X = \vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ if and only if, for all $s, s' \in X$ such that $\vec{t}_1\langle s\rangle = \vec{t}_1\langle s'\rangle$ there exists a $s'' \in X$ such that

$$\vec{t}_1\langle s''\rangle\vec{t}_2\langle s''\rangle = \vec{t}_1\langle s\rangle\vec{t}_2\langle s\rangle, \ \vec{t}_1\langle s''\rangle\vec{t}_3\langle s''\rangle = \vec{t}_1\langle s'\rangle\vec{t}_3\langle s'\rangle.$$

Independence Logic $\mathcal{I}$

$\mathcal{I}$ = First Order Logic + Independence Atoms

Properties of Independence Logic

Properties of Independence Logic (Grädel, Väänänen)

- Contains Dependence Logic;
- As expressive as Dependence Logic over sentences;
- More expressive on open formulas (no downwards closure).

Open Problem

What classes of teams are definable by open formulas in Independence Logic $\mathcal{I}$?

This talk will answer this.

Properties of Independence Logic

Properties of Independence Logic (Grädel, Väänänen)

- Contains Dependence Logic;
- As expressive as Dependence Logic over sentences;
- More expressive on open formulas (no downwards closure).

Open Problem

What classes of teams are definable by open formulas in Independence Logic $\mathcal{I}$?

This talk will answer this.

Outline

Inclusion Dependencies

Definition

R relation, $\vec{x}, \vec{y}$ tuples of attributes, $|\vec{x}| = |\vec{y}|$.
Then $R \models \vec{x} \subseteq \vec{y}$ if and only if for all $r \in R$ there exists an $r' \in R$ such that

$$r(\vec{x}) = r'(\vec{y}).$$

- Fairly well studied;
- Sound and complete axiomatization.

Example of Inclusion Dependency

Professor	University
Hilbert	Königsberg
Hilbert	Göttingen
Gauss	Göttingen

Person	Date of Birth
Hilbert	23/01/1862
Gauss	30/04/1777
Torvalds	28/12/1969

- $R \models \text{Professor} \subseteq \text{Person}$;
- $R \not\models \text{Person} \subseteq \text{Professor}$.

Example of Inclusion Dependency

Professor	University
Hilbert	Königsberg
Hilbert	Göttingen
Gauss	Göttingen

Person	Date of Birth
Hilbert	23/01/1862
Gauss	30/04/1777
Torvalds	28/12/1969

- $R \models \text{Professor} \subseteq \text{Person}$;
- $R \not\models \text{Person} \subseteq \text{Professor}$.

Exclusion Dependencies

Definition

R relation, $\vec{x}, \vec{y}$ tuples of attributes, $|\vec{x}| = |\vec{y}|$.
Then $R \models \vec{x} \mid \vec{y}$ if and only if, for all $r, r' \in R$,

$$r(\vec{x}) \neq r'(\vec{y}).$$

- Often, not used explicity;
- Very commonly used implicitly, for **typing** of attributes;
- Sound and complete axiomatization together with inclusion dependencies.

Example of Exclusion Dependency

Professor	University
Hilbert	Königsberg
Hilbert	Göttingen
Gauss	Göttingen

Person	Date of Birth
Hilbert	23/01/1862
Gauss	30/04/1777
Torvalds	28/12/1969

- $R \models$ University $|$ Date of Birth;
- $R \not\models$ Professor $|$ Person.

Example of Exclusion Dependency

Professor	University
Hilbert	Königsberg
Hilbert	Göttingen
Gauss	Göttingen

Person	Date of Birth
Hilbert	23/01/1862
Gauss	30/04/1777
Torvalds	28/12/1969

- $R \models$ University | Date of Birth;
- $R \not\models$ Professor | Person.

Inclusion and Exclusion Logic

Inclusion Atoms

$M \models_X \vec{t}_1 \subseteq \vec{t}_2$ if and only if $\{(\vec{t}_1\langle s\rangle, \vec{t}_2\langle s\rangle) : s \in X\} \models \vec{t}_1 \subseteq \vec{t}_2$;

Exclusion Atoms

$M \models_X \neg(\vec{t}_1 \mid \vec{t}_2)$ if and only if $\{(\vec{t}_1\langle s\rangle, \vec{t}_2\langle s\rangle) : s \in X\} \models \vec{t}_1 \mid \vec{t}_2$.

Inclusion/Exclusion Logic

I/E Logic = $\text{FO}_{\text{Team}}(\subseteq, \mid)$.
Inclusion Logic = only inclusion atoms,
Exclusion Logic = only exclusion atoms.

Direct Definitions for Tuple Existence Literals Semantics

Inclusion Atoms

$M \models_X \vec{t}_1 \subseteq \vec{t}_2$ if and only if for all $s \in X$ there exists a $s' \in X$ such that

$$\vec{t}_1\langle s\rangle = \vec{t}_2\langle s'\rangle;$$

Exclusion Atoms

$M \models_X \vec{t}_1 \mid \vec{t}_2$ if and only if, for all $s, s' \in X$,

$$\vec{t}_1\langle s\rangle \neq \vec{t}_2\langle s'\rangle.$$

Outline

Two Semantics for Disjuction

A lax semantics

$$M \models_X \psi_1 \vee^L \psi_2 \Leftrightarrow \exists Y, Z \text{ s.t. } X = Y \cup Z, M \models_Y \psi_1 \text{ and } M \models_Z \psi_2;$$

A strict semantics

$$M \models_X \psi_1 \vee^S \psi_2 \Leftrightarrow \exists Y, Z \text{ s.t. } X = Y \cup Z, X \cap Y = \emptyset, M \models_Y \psi_1 \text{ and } M \models_Z \psi_2;$$

$\mathcal{D}$ is usually given with $\vee^L$ (or even: $X \subseteq Y \cup Z$!).

Two Semantics for Disjuction

A lax semantics

$$M \models_X \psi_1 \vee^L \psi_2 \Leftrightarrow \exists Y, Z \text{ s.t. } X = Y \cup Z, M \models_Y \psi_1 \text{ and } M \models_Z \psi_2;$$

A strict semantics

$$M \models_X \psi_1 \vee^S \psi_2 \Leftrightarrow \exists Y, Z \text{ s.t. } X = Y \cup Z, X \cap Y = \emptyset, M \models_Y \psi_1 \text{ and } M \models_Z \psi_2;$$

$\mathcal{D}$ is usually given with $\vee^L$ (or even: $X \subseteq Y \cup Z$!).

In Dependence Logic, Lax = Strict

No difference for $\mathcal{D}$ (or for $\mathcal{T}^-$)

If $\psi_1, \psi_2 \in \mathcal{D}$, $M \models_X \psi_1 \vee^S \psi_2$ iff $M \models_X \psi_1 \vee^L \psi_2$.

Proof.

- If $M \models_X \psi_1 \vee^S \psi_2$, $M \models_X \psi_1 \vee^L \psi_2$;
- If $M \models_X \psi_1 \vee^L \psi_2$ then $X = X_1 \cup X_2$, $M \models_{X_1} \psi_1$, $M \models_{X_2} \psi_2$. Take $Y = X_2 \backslash X_1$: **by downwards closure**, $M \models_Y \psi_2$, $X_1 \cup Y = X$, so $M \models_X \psi_1 \vee^S \psi_2$.

□

In Inclusion Logic, Lax $\neq$ Strict

Different for Inclusion Logic!

There exist M, X and $\psi_1, \psi_2 \in FO(\subseteq)$ such that

$$M \models_X \psi_1 \vee^L \psi_2 \text{ but } M \not\models_X \psi_1 \vee^S \psi_2.$$

Proof.

Let $X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

and $\text{Dom}(M) = \{0 \ldots 4\}$. Then

$$M \models_X (x \subseteq y) \vee^L (y \subseteq z), \ M \not\models_X (x \subseteq y) \vee^S (y \subseteq z).$$

□

In Inclusion Logic, Lax $\neq$ Strict

Proof (continued).

$X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

- $M \models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $Y = \{s_0, s_1\}, Z = \{s_1, s_2\}$.
  $M \models_Y x \subseteq y, \quad M \models_Z y \subseteq z.$

In Inclusion Logic, Lax $\neq$ Strict

Proof (continued).

$X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

- $M \models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $Y = \{s_0, s_1\}, Z = \{s_1, s_2\}$.
  $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.

In Inclusion Logic, Lax $\neq$ Strict

Proof (continued).

$X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

- $M \models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $Y = \{s_0, s_1\}, Z = \{s_1, s_2\}$.
  $M \models_Y x \subseteq y, \quad M \models_Z y \subseteq z$.

□

In Inclusion Logic, Lax $\neq$ Strict

Proof (finished).

$X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

- $M \not\models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $X = Y \cup Z$, $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.
  $s_2 \notin Y$, so $s_2 \in Z$, so $s_1 \in Z$;
  $s_0 \notin Z$, so $s_0 \in Y$, so $s_1 \in Y$.
  So $Y \cap Z \neq \emptyset$.

In Inclusion Logic, Lax $\neq$ Strict

Proof (finished).

$X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

- $M \not\models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $X = Y \cup Z$, $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.
  $s_2 \notin Y$, so $s_2 \in Z$, so $s_1 \in Z$;
  $s_0 \notin Z$, so $s_0 \in Y$, so $s_1 \in Y$.
  So $Y \cap Z \neq \emptyset$.

□

In Inclusion Logic, Lax $\neq$ Strict

Proof (finished).

$X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

- $M \not\models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $X = Y \cup Z$, $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.
  $s_2 \notin Y$, so $s_2 \in Z$, so $s_1 \in Z$;
  $s_0 \notin Z$, so $s_0 \in Y$, so $s_1 \in Y$.
  So $Y \cap Z \neq \emptyset$.

□

In Inclusion Logic, Lax $\neq$ Strict

Proof (finished).

$X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

- $M \not\models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $X = Y \cup Z$, $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.
  $s_2 \notin Y$, so $s_2 \in Z$, so $s_1 \in Z$;
  $s_0 \notin Z$, so $s_0 \in Y$, so $s_1 \in Y$.
  So $Y \cap Z \neq \emptyset$.

In Inclusion Logic, Lax $\neq$ Strict

Proof (finished).

$X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

- $M \not\models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $X = Y \cup Z$, $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.
  $s_2 \notin Y$, so $s_2 \in Z$, so $s_1 \in Z$;
  $s_0 \notin Z$, so $s_0 \in Y$, so $s_1 \in Y$.

So $Y \cap Z \neq \emptyset$.

□

From Strict to Lax Disjunction

From strict to lax

If z not in ψ_1, ψ_2,

$$M \models_X \psi_1 \vee^L \psi_2 \Leftrightarrow M \models_X \forall z(\psi_1 \vee^S \psi_2).$$

Proof.

Let $0 \in \mathrm{Dom}(M)$, assume $|\mathrm{Dom}(M)| \geq 2$.
Suppose $X = Y \cup Z$, $M \models_Y \psi_1$, $M \models_Z \psi_2$, and let $W = Y \cap Z$.
Now define

$$Y' = (Y \backslash W)[M/z] \cup (W[0/z]), Z' = Z[M/z] \backslash Y'.$$

Then $Y' \cap Z' = \emptyset$, $Y' \cup Z' = X[M/z]$, $M \models_{Y'} \psi_1$, $M \models_{Z'} \psi_2$. □

Trivial Quantification and $\vee^S$

Corollary: $\vee^S$ is not invariant under trivial quantifications!

There exist formulas ψ_1 and $\psi_2 \in FO(\subseteq)$, such that z does not occur in ψ_1, ψ_2 but

$$\psi_1 \vee^S \psi_2 \not\equiv \forall z(\psi_1 \vee^S \psi_2).$$

Trivial Quantification and $\vee^L$

$\vee^L$ invariant under trivial quantification

For all ψ_1 and ψ_2 in $FO(\subseteq, \mid)$ and all $z \notin \psi_1, \psi_2$,

$$\psi_1 \vee^S \psi_2 \not\equiv \forall z(\psi_1 \vee^S \psi_2).$$

Proof.

Obvious from definition: if $X = Y \cup Z$, $M \models_Y \psi_1$, $M \models_Y \psi_2$, then $X[M/z] = Y[M/z] \cup Z[M/z]$, $M \models_{Y[M/z]} \psi_1$, $M \models_{Z[M/z]} \psi_2$. □

This strongly suggests that we want $\vee^L$ in our semantics.

Trivial Quantification and $\vee^L$

$\vee^L$ invariant under trivial quantification

For all ψ_1 and ψ_2 in $FO(\subseteq, \mid)$ and all $z \notin \psi_1, \psi_2$,

$$\psi_1 \vee^S \psi_2 \not\equiv \forall z(\psi_1 \vee^S \psi_2).$$

Proof.

Obvious from definition: if $X = Y \cup Z$, $M \models_Y \psi_1$, $M \models_Y \psi_2$, then $X[M/z] = Y[M/z] \cup Z[M/z]$, $M \models_{Y[M/z]} \psi_1$, $M \models_{Z[M/z]} \psi_2$. □

This strongly suggests that we want $\vee^L$ in our semantics.

Two Semantics for Existentials

A strict semantics

$$M \models_X \exists^S x\psi \Leftrightarrow \exists F : X \to M \text{ s.t. } M \models_{X[F/x]} \psi,$$

for $X[F/x] = \{s[F(s)/x] : s \in X\}$;

A lax semantics

$$M \models_X \exists^L x\psi \Leftrightarrow \exists F : H \to \mathcal{P}(M)\backslash\{\emptyset\} \text{ s.t. } M \models_{X[F/x]} \psi,$$

for $X[H/x] = \{s[m/x] : s \in X, m \in H(s)\}$.

$\mathcal{D}$ is usually given with $\exists^S$.

Two Semantics for Existentials

A strict semantics

$$M \models_X \exists^S x\psi \Leftrightarrow \exists F : X \to M \text{ s.t. } M \models_{X[F/x]} \psi,$$

for $X[F/x] = \{s[F(s)/x] : s \in X\}$;

A lax semantics

$$M \models_X \exists^L x\psi \Leftrightarrow \exists F : H \to \mathcal{P}(M)\backslash\{\emptyset\} \text{ s.t. } M \models_{X[F/x]} \psi,$$

for $X[H/x] = \{s[m/x] : s \in X, m \in H(s)\}$.

$\mathcal{D}$ is usually given with $\exists^S$.

In Dependence Logic, Strict $=$ Lax

No difference for $\mathcal{D}$

If $\psi \in \mathcal{D}$, $M \models_X \exists^S x\psi$ iff $M \models_X \exists^L x\psi$ (using AC).

Proof.

- If $M \models_X \exists^S x\psi$, $M \models_X \exists^L x\psi$;
- If $M \models_X \exists^L x\psi$, $M \models_{X[H/x]} \psi$ for some $H : X \to \mathcal{P}(M)\backslash\{\emptyset\}$. Let $F : X \to M$ be such that $F(s) \in H(s)$ for all $s \in X$: then $X[F/x] \subseteq X[H/x]$, so by downward closure $M \models_{X[F/x]} \psi$. Then $M \models_X \exists^S x\psi$, as required.

□

In Inclusion Logic, Strict $\neq$ Lax

Different for Inclusion Logic!

There exist M, X and $\psi \in FO(\subseteq)$ such that

$$M \models_X \exists^L x\psi \text{ but } M \not\models_X \exists^S \psi.$$

Proof.

Let $\mathrm{Dom}(M) = \{0, 1, 2\}$, $P^M = \{(0, 2), (1, 0), (1, 1)\}$, and $X = \{s_0, s_1\}$ for $s_0 = (y : 0)$, $s_1 = (y : 1)$.
Then
$M \models_X \exists^L x(y \subseteq x \wedge Pyx)$ but $M \not\models_X \exists^S x(y \subseteq x \wedge Pyx)$. □

In Inclusion Logic, Strict $\neq$ Lax

Proof (continued).

$\mathrm{Dom}(M) = \{0, 1, 2\}$, $P^M = \{(0, 2), (1, 0), (1, 1)\}$, and $X = \{s_0, s_1\}$ for $s_0 = (y : 0)$, $s_1 = (y : 1)$.

- $M \models_X \exists^L x(y \subseteq x \wedge Pyx)$: let $H : X \to \mathcal{P}(M)$ be such that $H(s_0) = \{2\}$, $H(s_1) = \{0, 1\}$. Then

$$X[H/x] = \begin{array}{c|c|c} & y & x \\ \hline s'_0 & 0 & 2 \\ s'_1 & 1 & 0 \\ s'_2 & 1 & 1. \end{array}$$

and this team satisfies $y \subseteq x$ and Pyx.

□

In Inclusion Logic, Strict $\neq$ Lax

Proof (finished).

$\mathrm{Dom}(M) = \{0, 1, 2\}$, $P^M = \{(0, 2), (1, 0), (1, 1)\}$, and $X = \{s_0, s_1\}$ for $s_0 = (y : 0)$, $s_1 = (y : 1)$.

- $M \not\models_X \exists^S x(y \subseteq x \wedge Pyx)$: take any $F : X \to M$, and consider $X[F/x]$.
  If $F(s_0) \neq 2$, $M \not\models_{X[F/x]} Pyx$; so $F(s_0) = 2$.
  But then

$$X[F/x] = \begin{array}{c|c|c} & y & x \\ \hline s'_0 & 0 & 2 \\ s'_1 & 1 & F(s_1) \end{array}$$

  and $M \not\models_{X[F/x]} y \subseteq x$, since $F(s_1) \neq 0$ or $F(s_1) \neq 1$.

□

From Strict to Lax Existentials

From strict to lax semantics

If z not in ψ and $z \neq x$,

$$M \models_X \exists^L x\psi \Leftrightarrow M \models_X \forall z \exists^L x\psi.$$

From Strict to Lax Existentials

Proof.

Suppose that for $H : X \to \mathcal{P}(X) \backslash \{\emptyset\}$, $M \models_{X[H/x]} \psi$.
For every $s \in X$, let $m_s \in H(s)$; then define $F : X[M/z] \to M$ as

$$F(s[m/z]) = \begin{cases} m & \text{if } m \in H(s); \\ m_s & \text{otherwise.} \end{cases}$$

Forgetting the variable z, $X[M/z][F/x]$ is precisely $X[H/z]$; hence,
$M \models_{X[M/z][F/x]} \psi$, as required (other direction is trivial). □

Trivial Quantification and $\exists^S$

Corollary: $\exists^S$ is not invariant under trivial quantifications!

There exists a $\psi \in FO(\subseteq)$, such that z does not occur in it but

$$\exists^S x\psi \not\equiv \forall z \exists^S x\psi.$$

Trivial Quantification and $\exists^L$

$\exists^L$ invariant under trivial quantification

For all ψ in $FO(\subseteq, \mid)$ and all $z \notin \psi$,

$$\exists^L x \psi \equiv \forall z \exists^L s \psi.$$

Proof.

If for $H : X[M/z] \to \mathcal{P}(M)$ it holds that $M \models_{X[M/z][H/x]} \psi$, define $H' : X \to \mathcal{P}(M)$ as

$$H'(s) = \{m \in M : \exists m' \in M \text{ s.t. } m \in H(s[m'/z])\}.$$

Then $M \models_{X[H'/z]} \psi$, as required. □

This strongly suggests that we want $\exists^L$ in our semantics.

Trivial Quantification and $\exists^L$

$\exists^L$ invariant under trivial quantification

For all ψ in $FO(\subseteq, \mid)$ and all $z \notin \psi$,

$$\exists^L x\psi \equiv \forall z \exists^L s\psi.$$

Proof.

If for $H : X[M/z] \to \mathcal{P}(M)$ it holds that $M \models_{X[M/z][H/x]} \psi$, define $H' : X \to \mathcal{P}(M)$ as

$$H'(s) = \{m \in M : \exists m' \in M \text{ s.t. } m \in H(s[m'/z])\}.$$

Then $M \models_{X[H'/z]} \psi$, as required. □

This strongly suggests that we want $\exists^L$ in our semantics.

Outline

1. Non-Functional Dependencies
   - Independence Atoms
   - Inclusion and Exclusion Atoms
2. Semantics
   - Strict and Lax Operators
   - Game Theoretic Semantics
3. Expressivity
   - Exclusion Logic
   - Inclusion/Exclusion Logic
4. Definability in I/E Logic

GTS for Dependence Logic

GTS (Väänänen 07)

For every model M, team X and formula ϕ with free variables in Dom(X) one can define an imperfect information, zero-sum two-player game $G^M_X(\phi)$.

Theorem (Väänänen 07)

$M \models_X \phi \Leftrightarrow$ Player II has a uniform winning strategy in $G^M_X(\phi)$.

Can we find a similar game for Inclusion/Exclusion Logic?

GTS for Dependence Logic

GTS (Väänänen 07)

For every model M, team X and formula ϕ with free variables in Dom(X) one can define an imperfect information, zero-sum two-player game $G^M_X(\phi)$.

Theorem (Väänänen 07)

$M \models_X \phi \Leftrightarrow$ Player*II* has a uniform winning strategy in $G^M_X(\phi)$.

Can we find a similar game for Inclusion/Exclusion Logic?

The games $G^M_X(\phi)$

The game $G^M_X(\phi)$ for I/E Logic

Let M, X ϕ as before ($\phi \in$I/E). Define $G^M_X(\phi)$ as follows:

- Initial positions = $\{(\phi, s) : s \in X\}$;
- Given a position p, its successor set Succ(p) is
  1. $\{(\theta_1, s), (\theta_2, s)\}$ if $p = (\theta_1 \vee \theta_2, s)$ or $(\theta_1 \wedge \theta_2, s)$;
  2. $\{(\theta, s[m/x]) : m \in \mathrm{Dom}(M)\}$ if $p = (\exists x \theta, s)$ or $(\forall x \theta, s)$;
- Given a position p, the *active player* $T(p)$ is
  1. *I* if p is $(\theta_1 \wedge \theta_2, s)$ or $(\forall x \theta, s)$;
  2. *II* if p is $(\theta_1 \vee \theta_2, s)$ or $(\exists x \theta, s)$.
- If $p = (\vec{t}_1 \subseteq \vec{t}_2, s)$ or $(\vec{t}_1 \mid \vec{t}_2, s)$ then p winning for *II*;
- If $p = (\alpha, s)$, α FO literal, p winning for *II* iff $M \models_s \alpha$.

Plays

Plays

A play of $G^M_X(\phi)$ is a sequence of positions $p_1 \ldots p_n$ s.t.

- p_1 is initial;
- $p_{i+1} \in \text{Succ}(p_i)$ $(i = 1 \ldots n-1)$.

Complete Plays

A play $p_1 \ldots p_n$ is *complete* iff p_n is terminal.

Winning Plays

A play $p_1 \ldots p_n$ is winning (for II) iff p_n is winning (for II).

Strategies

Strategies

A strategy τ (for II) for $G^M_X(\phi)$ is a function from positions p with $T(p) = II$ to $\mathcal{P}(\text{Succ}(p)) \backslash \emptyset$.

Deterministic Strategies

A strategy τ is *deterministic* if $\tau(p)$ is always a singleton.

Play following a strategy

A play $p_1 \ldots p_n$ *follows* τ if

$$T(p_i) = II \Rightarrow p_{i+1} \in \tau(p_i).$$

Winning Strategies

$P(G^M_X(\phi), \tau)$

$P(G^M_X(\phi), \tau) = \{\vec{p} : \vec{p}$ play of $G^M_X(\phi)$, Player *II* follows τ in $\vec{p}\}$.

Winning Strategy

A strategy τ is *winning* iff

$$\vec{p} \text{ complete}, \vec{p} \in P(G^M_X(\phi), \tau) \Rightarrow \vec{p} \text{ winning}.$$

Uniformity

Uniform Strategy

A strategy τ is uniform iff, for all $p_1 \ldots p_n = \vec{p} \in P(G^M_X(\phi), \tau)$,

- If p_n is $(\vec{t}_1 \subseteq \vec{t}_2, s)$ then $\exists q_1 \ldots q_{n'} = \vec{q} \in P(G^M_X(\phi), \tau)$ s.t.
  1. $q_n = (\vec{t}_1 \subseteq \vec{t}_2, s')$ for the same instance of the atom;
  2. $t_1\langle s\rangle = t_2\langle s'\rangle$;
- If p_n is $(\vec{t}_1 \mid \vec{t}_2, s)$ then $\neg\exists q_1 \ldots q_{n'} = \vec{q} \in P(G^M_X(\phi), \tau)$ s.t.
  1. $q_n = (\vec{t}_1 \mid \vec{t}_2, s')$ for the same instance of the atom;
  2. $t_1\langle s\rangle = t_2\langle s'\rangle$;

Equivalence

Lax Semantics and GTS

For all suitable M, X, ϕ,

$$M \models_X \phi \text{ (Lax) } \Leftrightarrow \exists \text{ u.w.s. for } II \text{ in } G^M_X(\phi);$$

Strict Semantics and GTS

For all suitable M, X, ϕ,

$$M \models_X \phi \text{ (Strict) } \Leftrightarrow \exists \text{ deterministic u.w.s. for } II \text{ in } G^M_X(\phi);$$

Outline

From Exclusion to Dependence

Dependence atoms in Exclusion Logic

The dependence atom $=(t_1 \ldots t_n)$ is equivalent to the expression

$$\forall z(z = t_n \vee t_1 \ldots t_{n-1} z \mid t_1 \ldots t_{n-1} t_n).$$

From Exclusion to Dependence

Dependence atoms in Exclusion Logic (simple case)

The dependence atom $=(x, y)$ is equivalent to the expression

$$\forall z(z = y \vee xz \mid xy).$$

Proof (Left to Right).

Suppose $M \models_X =(x, y)$, let $Y = \{s[m/z] : s \in X, m \neq s(y)\}$.
If $M \models_Y xz \mid xy$, done.
So take $h, h' \in Y$, $h(x) = h'(x)$, $h'(y) = h(z) \neq h(y)$.
Contradiction. □

From Exclusion to Dependence

Dependence atoms in Exclusion Logic (simple case)

The dependence atom $=\!(x, y)$ is equivalent to the expression

$$\forall z(z = y \vee xz \mid xy).$$

Proof (Right to Left).

Suppose $M \not\models_X =\!(x, y)$. Then exist $s, s' \in X$ s.t. $s(x) = s'(x)$, $s(y) \neq s'(y)$.
Consider $h = s[s'(y)/z]$, $h' = s'[s(y)/z]$.
$h(y) \neq h(z)$, $h'(y) \neq h'(z)$.
But $h(x) = s(x) = s'(x) = h'(x)$ and $h(z) = s'(y) = h'(y)$.
So $M \not\models_X \forall z(z = y \vee xz \mid xy)$. □

From Dependence to Exclusion

Exclusion atoms in $\mathcal{D}$

There exists a formula ϕ in Dependence Logic such that

$$M \models_X \phi \text{ if and only if } M \models_X \vec{t}_1 \mid \vec{t}_2$$

Proof.

$\vec{t}_1 \mid \vec{t}_2$ holds of the empty team, and $M \models_X \vec{t}_1 \mid \vec{t}_2$ iff

$$M, \mathrm{Rel}(X) \models \forall \vec{s}_1 \vec{s}_2 (R\vec{s}_1 \wedge R\vec{s}_2 \rightarrow \vec{t}_1\langle \vec{s}_1 \rangle \neq \vec{t}_2 \langle \vec{s}_2 \rangle).$$

By KV 2009, this is expressible in Dependence Logic. □

Exclusion Logic and Dependence Logic

Corollary

Exclusion Logic and Dependence Logic are equivalent.

Even wrt open formulas!

Outline

1. Non-Functional Dependencies
   - Independence Atoms
   - Inclusion and Exclusion Atoms
2. Semantics
   - Strict and Lax Operators
   - Game Theoretic Semantics
3. Expressivity
   - Exclusion Logic
   - Inclusion/Exclusion Logic
4. Definability in I/E Logic

I/E Logic and Independence Logic

Independence atoms in I/E Logic

$\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ is equivalent to

$$\forall \vec{p}_1 \vec{p}_2 \vec{p}_3 ((\vec{p}_1 \vec{p}_2 \mid \vec{t}_1 \vec{t}_2) \vee_{\vec{p}_1 \vec{p}_2 \vec{p}_3} (\vec{p}_1 \vec{p}_3 \mid \vec{t}_1 \vec{t}_3) \vee_{\vec{p}_1 \vec{p}_2 \vec{p}_3} \vee_{\vec{p}_1 \vec{p}_2 \vec{p}_3} \vec{p}_1 \vec{p}_2 \vec{p}_3 \subseteq \vec{t}_1 \vec{t}_2 \vec{t}_3).$$

Inclusion Atoms in $\mathcal{I}$

$\vec{t}_1 \subseteq \vec{t}_2$ is equivalent to

$$\forall u_1 u_2 \vec{z} ((\vec{z} \neq \vec{t}_1 \wedge \vec{z} \neq \vec{t}_2) \vee (u_1 \neq u_2 \wedge \vec{z} \neq \vec{t}_2) \vee \vee ((u_1 = u_2 \vee \vec{z} = \vec{t}_2) \wedge \vec{z} \perp_{\emptyset} u_1 u_2)).$$

Tuple Existence Logic and Independence Logic

Independence Logic is I/E Logic

- For every formula $\phi \in \mathcal{I}$ there exists a ψ of I/E Logic s.t.

$$M \models_X \phi \Leftrightarrow M \models_X \psi;$$

- For every formula ψ of I/E Logic there exists a $\phi \in \mathcal{I}$ s.t.

$$M \models_X \psi \Leftrightarrow M \models_X \phi.$$

Backslashed disjunction

Backslashed disjunction

V finite set of variables, $\phi \vee_V \psi$ equivalent to

$$\exists z_1 z_2 (=(V, z_2) \wedge =(V, z_2) \wedge ((z_1 = z_2 \wedge \phi) \vee (z_1 \neq z_2 \wedge \psi)))$$

- Expressible in I/E Logic (dep atom expressible).
- $M \models_X \phi \vee_V \psi \Leftrightarrow \exists YZ$ s.t.
  1. $X = Y \cup Z$;
  2. $M \models_Y \phi$, $M \models_Z \psi$;
  3. For all $s, s' \in X$ s.t. $s \equiv_V s'$,
     - $s \in Y \Leftrightarrow s' \in Y$,
     - $s \in Z \Leftrightarrow s' \in Z$.

Independence Atoms in I/E Logic

Independence atoms in I/E

$\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ is equivalent to

$$\forall \vec{p}_1 \vec{p}_2 \vec{p}_3 ((\vec{p}_1 \vec{p}_2 \mid \vec{t}_1 \vec{t}_2) \vee_{\vec{p}_1 \vec{p}_2 \vec{p}_3} (\vec{p}_1 \vec{p}_3 \mid \vec{t}_1 \vec{t}_3) \vee_{\vec{p}_1 \vec{p}_2 \vec{p}_3}$$
$$\vee_{\vec{p}_1 \vec{p}_2 \vec{p}_3} \vec{p}_1 \vec{p}_2 \vec{p}_3 \subseteq \vec{t}_1 \vec{t}_2 \vec{t}_3).$$

Independence Atoms in I/E Logic

Independence atoms in I/E (simple case)

$y \perp_x z$ is equivalent to the expression

$$\forall p_1 p_2 p_3 ((p_1 p_2 \mid xy) \vee_{\vec{p}} (p_1 p_3 \mid xz) \vee_{\vec{p}} p_1 p_2 p_3 \subseteq xyz).$$

Proof (Left to Right).

Suppose $M \models_X y \perp_x z$, let $h \in X[M/p_1 p_2 p_3]$.

- If $\forall s \in X$, $s(xy) \neq h(p_1 p_2)$, $h \in Y_1$: $M \models_{Y_1} (p_1 p_2 \mid xy)$;
- If $\forall s \in X$, $s(xz) \neq h(p_1 p_3)$, $h \in Y_2$: $M \models_{Y_2} (p_1 p_3 \mid xz)$.
- Otherwise, $h \in Y_3$: $M \models_{Y_3} (p_1 p_2 p_3 \subseteq xyz)$.

□

Independence Atoms in I/E Logic

Independence atoms in I/E (simple case)

$y \perp_x z$ is equivalent to the expression

$$\forall p_1 p_2 p_3 ((p_1 p_2 \mid xy) \vee_{\vec{p}} (p_1 p_3 \mid xz) \vee_{\vec{p}} p_1 p_2 p_3 \subseteq xyz).$$

Proof (Left to Right).

Suppose $M \models_X y \perp_x z$, let $h \in X[M/p_1 p_2 p_3]$.

- If $\forall s \in X$, $s(xy) \neq h(p_1 p_2)$, $h \in Y_1$: $M \models_{Y_1} (p_1 p_2 \mid xy)$;
- If $\forall s \in X$, $s(xz) \neq h(p_1 p_3)$, $h \in Y_2$: $M \models_{Y_2} (p_1 p_3 \mid xz)$.
- Otherwise, $h \in Y_3$: $M \models_{Y_3} (p_1 p_2 p_3 \subseteq xyz)$.

□

Independence Atoms in I/E Logic

Independence atoms in I/E (simple case)

$y \perp_x z$ is equivalent to the expression

$$\forall p_1 p_2 p_3((p_1 p_2 \mid xy) \vee_{\vec{p}} (p_1 p_3 \mid xz) \vee_{\vec{p}} p_1 p_2 p_3 \subseteq xyz).$$

Proof (Left to Right).

Suppose $M \models_X y \perp_x z$, let $h \in X[M/p_1 p_2 p_3]$.

- If $\forall s \in X$, $s(xy) \neq h(p_1 p_2)$, $h \in Y_1$: $M \models_{Y_1} (p_1 p_2 \mid xy)$;
- If $\forall s \in X$, $s(xz) \neq h(p_1 p_3)$, $h \in Y_2$: $M \models_{Y_2} (p_1 p_3 \mid xz)$.
- Otherwise, $h \in Y_3$: $M \models_{Y_3} (p_1 p_2 p_3 \subseteq xyz)$.

Independence Atoms in I/E Logic

Independence atoms in I/E (simple case)

$y \perp_x z$ is equivalent to the expression

$$\forall p_1 p_2 p_3((p_1 p_2 \mid xy) \vee_{\vec{p}} (p_1 p_3 \mid xz) \vee_{\vec{p}} p_1 p_2 p_3 \subseteq xyz).$$

Proof (Left to Right).

Suppose $M \models_X y \perp_x z$, let $h \in X[M/p_1 p_2 p_3]$.

- If $\forall s \in X, s(xy) \neq h(p_1 p_2)$, $h \in Y_1$: $M \models_{Y_1} (p_1 p_2 \mid xy)$;
- If $\forall s \in X, s(xz) \neq h(p_1 p_3)$, $h \in Y_2$: $M \models_{Y_2} (p_1 p_3 \mid xz)$.
- Otherwise, $h \in Y_3$: $M \models_{Y_3} (p_1 p_2 p_3 \subseteq xyz)$.

$\square$

Independence Atoms in I/E Logic

Proof (Right to Left).

Suppose $M \not\models_X y \perp_x z$: $\exists s, s' \in X$ s.t $s(x) = s'(x)$, but
$s'' \in X \Rightarrow s''(xy) \neq s(xy)$ or $s''(xz) \neq s'(xz)$.

$$m_1 = s(x) = s'(x),\ m_2 = s(y),\ m_3 = s'(z).$$

$$h = s[m_1/p_1][m_2/p_2][m_3/p_3],\ h = s[m_1/p_1][m_2/p_2][m_3/p_3].$$

1. $h, h' \in Y_1$, $M \models_{Y_1} p_1 p_2 \mid xy$: NO, $h(xy) = h(p_1 p_2)$;
2. $h, h' \in Y_2$, $M \models_{Y_2} p_1 p_3 \mid xz$: NO, $h'(xz) = h'(p_1 p_3)$;
3. $h, h' \in Y_3$, $M \models_{Y_3} p_1 p_2 p_3 \subseteq xyz$: NO, contradiction.

□

Independence Atoms in I/E Logic

Proof (Right to Left).

Suppose $M \not\models_X y \perp_x z$: $\exists s, s' \in X$ s.t $s(x) = s'(x)$, but
$s'' \in X \Rightarrow s''(xy) \neq s(xy)$ or $s''(xz) \neq s'(xz)$.

$$m_1 = s(x) = s'(x),\ m_2 = s(y),\ m_3 = s'(z).$$

$$h = s[m_1/p_1][m_2/p_2][m_3/p_3],\ h = s[m_1/p_1][m_2/p_2][m_3/p_3].$$

1. $h, h' \in Y_1$, $M \models_{Y_1} p_1p_2 \mid xy$: NO, $h(xy) = h(p_1p_2)$;
2. $h, h' \in Y_2$, $M \models_{Y_2} p_1p_3 \mid xz$: NO, $h'(xz) = h'(p_1p_3)$;
3. $h, h' \in Y_3$, $M \models_{Y_3} p_1p_2p_3 \subseteq xyz$: NO, contradiction.

□

Inclusion Atoms in $\mathcal{I}$

Inclusion atoms in $\mathcal{I}$

$\vec{t}_1 \subseteq \vec{t}_2$ is equivalent to

$$\forall u_1 u_2 \vec{z}((\vec{z} \neq \vec{t}_1 \wedge \vec{z} \neq \vec{t}_2) \vee (u_1 \neq u_2 \wedge \vec{z} \neq \vec{t}_2) \vee \vee ((u_1 = u_2 \vee \vec{z} = \vec{t}_2) \wedge \vec{z} \perp_{\emptyset} u_1 u_2)).$$

Inclusion Atoms in $\mathcal{I}$

Inclusion atoms in $\mathcal{I}$ (simple case)

$$x \subseteq y \equiv \forall u_1 u_2 z((z \neq x \wedge z \neq y) \vee (u_1 \neq u_2 \wedge z \neq y) \vee \\ \vee ((u_1 = u_2 \vee z = y) \wedge z \perp_{\emptyset} u_1 u_2)).$$

Proof (Left to Right).

$Y = \{s[m_1/u_1][m_2/u_2][m_3/z] : s \in X, m_1 = m_2$ and and $m_3 \in \{s(x), s(y)\}$, or $m_1 \neq m_2$ and $m_3 = s(y)\}$.

If I show that $Y \models z \perp_{\emptyset} u_1 u_2$, done. Take $s, s' \in Y$.

If $s(z) = s(y)$, $s[s'(u_1)/u_1][s'(u_2)/u_2] \in Y$;

If $s(z) = s(x), \exists s'' \in X$, $s''(y) = s(x)$;

Then $s''[s'(u_1)/u_1][s'(u_2)/u_2] \in Y$, done. □

Inclusion Atoms in $\mathcal{I}$

Inclusion atoms in $\mathcal{I}$ (simple case)

$$x \subseteq y \equiv \forall u_1 u_2 z((z \neq x \land z \neq y) \lor (u_1 \neq u_2 \land z \neq y) \lor \\ \lor ((u_1 = u_2 \lor z = y) \land z \perp_\emptyset u_1 u_2)).$$

Proof (Right to Left).

$s \in X$, $h = s[0/u_1][0/u_2][s(x)/z]$, $h' = s[0/u_1][1/u_2][s(y)/z]$.
$h, h' \in Y$, $Y \models z \perp_\emptyset u_1 u_2$?
Then $\exists h''$, $h''(u_1) = 0$, $h''(u_2) = 1$, $h''(z) = h(z) = s(x)$.
But then $h''(y) = h''(z) = s(x)$. □

Definability in I/E Logic

From I/E Logic to Σ_1^1

For every formula $\phi \in$ I/E there exists a sentence $\phi' \in \Sigma_1^1$ such that $M \models_X \phi$ if and only if $M, \text{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

From Σ_1^1 to I/E Logic

For every sentence $\phi'(R) \in \Sigma_1^1$ there exists a formula $\phi \in$ I/E such that $M \models_X \phi$ if and only if $M, \text{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Thanks to Juha Kontinen for pointing out this requirement!

Definability in I/E Logic

From I/E Logic to Σ^1_1

For every formula $\phi \in$ I/E there exists a sentence $\phi' \in \Sigma^1_1$ such that $M \models_X \phi$ if and only if $M, \mathrm{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

From Σ^1_1 to I/E Logic

For every sentence $\phi'(R) \in \Sigma^1_1$ there exists a formula $\phi \in$ I/E such that $M \models_X \phi$ if and only if $M, \mathrm{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Thanks to Juha Kontinen for pointing out this requirement!

Corollary: Definability on Independence Logic

From Independence Logic to Σ_1^1

For every formula $\phi \in \mathcal{I}$ there exists a sentence $\phi' \in \Sigma_1^1$ such that $M \models_X \phi$ if and only if $M, \text{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

From Σ_1^1 to Independence Logic

For every sentence $\phi'(R) \in \Sigma_1^1$ there exists a formula $\phi \in \mathcal{I}$ such that $M \models_X \phi$ if and only if $M, \text{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Left to Right

From I/E Logic to Σ_1^1

For every formula $\phi \in$ I/E there exists a sentence $\phi' \in \Sigma_1^1$ such that $M \models_X \phi$ if and only if $M, \mathrm{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Proof.

By structural induction over ϕ (easy). □

Left to Right

From I/E Logic to Σ^1_1

For every formula $\phi \in$ I/E there exists a sentence $\phi' \in \Sigma^1_1$ such that $M \models_X \phi$ if and only if $M, \mathrm{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Proof.

By structural induction over ϕ (easy). □

Right to Left

From Σ^1_1 to I/E Logic

For every sentence $\phi'(R) \in \Sigma^1_1$ there exists a formula $\phi \in$ I/E such that $M \models_X \phi$ if and only if $M, \mathrm{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Proof.

Similar to the ones in KV 2009 and KN 2009.
Write $\phi'(R)$ as $\exists R' \exists \vec{f} \forall \vec{z}((R'\vec{x} \leftrightarrow R\vec{x}) \wedge \psi(R', \vec{z}))$ where $\vec{x}$ subsequence of $\vec{z}$,
ψ quantifier free, R not in ψ, each f_i only as $f_i(\vec{w}_i)$ for some fixed $\vec{w}_i \subseteq \vec{z}$, R' only as $R'\vec{x}$. □

Right to Left

Proof (continued).

Write $\phi'(R)$ as $\exists R' \exists \vec{f}\, \forall \vec{z}((R'\vec{x} \leftrightarrow R\vec{x}) \wedge \psi(R', \vec{z}))$ where $\vec{x}$ subsequence of $\vec{z}$,
ψ quantifier free, R not in ψ, each f_i only as $f_i(\vec{w}_i)$ for some fixed $\vec{w}_i \subseteq \vec{z}$, R' only as $R'\vec{x}$.
Then $M, \mathrm{Rel}(X) \models \phi'$ if and only if

$$M, \mathrm{Rel}(X) \models \exists g_1 g_2 \exists \vec{f}\, \forall \vec{z}((f_1(\vec{x}) = f_2(\vec{x}) \leftrightarrow R\vec{x}) \wedge \psi'(\vec{z}))$$

where $\psi' = \psi[f_1\vec{x} = f_2\vec{x} / R\vec{x}]$. □

Right to Left

Proof (continued).

$$\phi' \equiv \exists g_1 g_2 \exists \vec{f} \forall \vec{z}((g_1(\vec{x}) = g_2(\vec{x}) \leftrightarrow R\vec{x}) \wedge \psi'(\vec{z}))$$

where $\psi' = \psi[g_1\vec{x} = g_2\vec{x}/R\vec{x}]$.
Then, if X nonempty, $\text{Dom}(X) = \vec{y}$, $M, \text{Rel}(X) \models \phi'$ iff

$$M \models_X \forall \vec{z} \exists u_1 u_2 \vec{v}\left(\left(\bigwedge_{i=1}^{2} =(\vec{x}, u_i) \wedge \bigwedge_j =(\vec{w}_j, v_j) \right) \wedge \right.$$
$$\left. \wedge ((\vec{x} \subseteq \vec{y} \wedge u_1 = u_2) \vee (\vec{x} \mid \vec{y} \wedge u_1 \neq u_2)) \wedge \theta \right)$$

where θ is $\psi'[u_1/g_1\vec{x}][u_2/g_2\vec{x}][\vec{w}/\vec{f}\vec{w}]$. □

Right to Left

Proof (continued).

Suppose that, for all s with domain $\vec{z}$,

$$M, \mathrm{Rel}(X), g_1, g_2, \vec{f} \models (g_1(\vec{x}) = g_2(\vec{x}) \leftrightarrow R\vec{x}) \wedge \psi'(\vec{z}).$$

Extend X to Y choosing the u_1, u_2, $\vec{v}$ according to g_1, g_2, $\vec{f}$.

- $M \models_Y \bigwedge_{i=1}^{2} {=}(\vec{x}, u_i) \wedge \bigwedge_j {=}(\vec{w}_j, v_j)$: obvious;
- $M \models_Y \theta$: by construction;
- $M \models_Y (\vec{x} \subseteq \vec{y} \wedge u_1 = u_2) \vee (\vec{x} \mid \vec{y} \wedge u_1 \neq u_2)$:
  If $u_1 = u_2$, $\vec{x} \in \mathrm{Rel}(X)$, so $\vec{x} \subseteq \vec{y}$;
  If $u_1 \neq u_2$, $\vec{x} \notin \mathrm{Rel}(X)$, so $\vec{x} \mid \vec{y}$.

□

Right to Left

Proof (continued).

Conv., suppose X nonempty, $Y = X[M/\vec{z}][G_1/u_1][G_2/u_2][\vec{F}/\vec{v}]$,

$$M \models_Y \bigwedge_{i=1}^{2} =\!(\vec{x}, u_i) \wedge \bigwedge_j =\!(\vec{w}_j, v_j),$$

$$M \models_Y (\vec{x} \subseteq \vec{y} \wedge u_1 = u_2) \vee (\vec{x} \mid \vec{y} \wedge u_1 \neq u_2),$$

$$M \models_Y \theta.$$

Choose $g_1(\vec{x}), g_2(\vec{x}), \vec{f}(\vec{w})$ according to $G_1, G_2, \vec{F}$.
Let s be any assignment, domain = $\vec{z}$. □

Right to Left

Proof (continued).

Choose $g_1(\vec{x}), g_2(\vec{x}), \vec{f}(\vec{w})$ according to $G_1, G_2, \vec{F}$.
Let s be any assignment, domain = $\vec{z}$.

- $M, \text{Rel}(R), g_1, g_2, \vec{f} \models_s \psi'$: Take $h \in X$. Then $h[s/\vec{z}][g_1/u_1][g_2/u_2][\vec{f}/\vec{v}] \in Y$, $M \models_Y \theta$.
- $M, \text{Rel}(R), g_1, g_2, \vec{f} \models_s g_1(\vec{x}) = g_2(\vec{x}) \leftrightarrow R\vec{x}$:
  Suppose $g_1(\vec{x}) = g_2(\vec{x})$, let $h \in X$.
  Consider $o = h[s/\vec{z}][g_1/u_1][g_2/u_2][\vec{f}/\vec{v}]$:
  $o \in Y_1, M \models_{Y_1} \vec{x} \subseteq \vec{y}$. So $\exists o' \in Y_1$, $o'(\vec{y}) = o(\vec{x})$, so $s(\vec{x}) = o(\vec{x}) \in \text{Rel}(X)$.

□

Right to Left

Proof (finished).

Choose $g_1(\vec{x}), g_2(\vec{x}), \vec{f}(\vec{w})$ according to $G_1, G_2, \vec{F}$.
Let s be any assignment, domain = $\vec{z}$.

- $M, \text{Rel}(R), g_1, g_2, \vec{f} \models_s g_1(\vec{x}) = g_2(\vec{x}) \leftrightarrow R\vec{x}$:
  Suppose $g_1(\vec{x}) \neq g_2(\vec{x})$, let $h \in X$.
  Consider $o = h[s/\vec{z}][g_1/u_1][g_2/u_2][\vec{f}/\vec{v}]$:
  $o \in Y_2, M \models_{Y_2} \vec{x} \mid \vec{y}$. So $\forall o' \in Y_2$, $o'(\vec{y}) \neq o(\vec{x})$.
  But for all $h' \in X$, $o' = h'[s/\vec{z}][g_1/u_1][g_2/u_2][\vec{f}/\vec{v}] \in Y_2$;
  then, for all such h',
  $s(\vec{x}) = o(\vec{x}) \neq o'(\vec{y}) = h'(\vec{y})$.
  Therefore, $s(\vec{x}) \notin \text{Rel}(X)$.

□