Independence logic and tuple existence atoms, part 2

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Outline

Independence Logic

Independence Atoms (Grädel, Väänänen)

$M \models_X = \vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ if and only if, for all $s, s' \in X$ such that $\vec{t}_1\langle s\rangle = \vec{t}_1\langle s'\rangle$ there exists a $s'' \in X$ such that

$$\vec{t}_1\langle s''\rangle\vec{t}_2\langle s''\rangle = \vec{t}_1\langle s\rangle\vec{t}_2\langle s\rangle, \; \vec{t}_1\langle s''\rangle\vec{t}_3\langle s''\rangle = \vec{t}_1\langle s'\rangle\vec{t}_3\langle s'\rangle.$$

Independence Logic $\mathcal{I}$

$\mathcal{I}$ = First Order Logic + Independence Atoms

Properties of Independence Logic

Properties of Independence Logic (Grädel, Väänänen)

- Contains Dependence Logic;
- As expressive as Dependence Logic over sentences;
- More expressive on open formulas (no downwards closure).

Open Problem

What classes of teams are definable by open formulas in Independence Logic $\mathcal{I}$?

This talk will answer this.

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Tuple Existence Logic

Tuple Existence Atoms (Inclusion Dependencies)

$M \models_X = \vec{t}_1 @ \vec{t}_2$ if and only if, for all $s \in X$ there exists a $s' \in X$ such that

$$\vec{t}_1\langle s \rangle = \vec{t}_2\langle s' \rangle;$$

Negated Tuple Existence Atoms (Exclusion Dependencies)

$M \models_X = \neg(\vec{t}_1 @ \vec{t}_2)$ if and only if, for all $s, s' \in X$,

$$\vec{t}_1\langle s \rangle \neq \vec{t}_2\langle s' \rangle.$$

Tuple Existence Logic

Tuple Existence Logic $\mathcal{T}$

- $\mathcal{T}^+$ = First Order Logic + Inclusion atoms $\vec{t}_1 \circledcirc \vec{t}_2$;
- $\mathcal{T}^-$ = First Order Logic + Exclusion atoms $\neg(\vec{t}_1 \circledcirc \vec{t}_2)$;
- $\mathcal{T}$ = First Order Logic + Inclusion and Exclusion atoms.

Exclusion Logic and Dependence Logic

Dependence atoms in $\mathcal{T}^-$

The dependence atom $=\!(t_1 \ldots t_n)$ is equivalent to the expression

$$\forall z (z = t_n \vee \neg(t_1 \ldots t_{n-1} z \circledcirc t_1 \ldots t_{n-1} t_n)).$$

Exclusion atoms in $\mathcal{D}$

There exists a formula ϕ in Dependence Logic such that

$$M \models_X \phi \text{ if and only if } M \models_X \neg(\vec{t}_1 \circledcirc \vec{t}_2)$$

Exclusion Logic and Dependence Logic

Dependence Logic is Exclusion Logic

- For every formula $\phi \in \mathcal{D}$ there exists a $\psi \in \mathcal{T}^-$ such that
$$M \models_X \phi \Leftrightarrow M \models_X \psi;$$
- For every formula $\psi \in \mathcal{T}^-$ there exists a $\phi \in \mathcal{D}$ such that
$$M \models_X \psi \Leftrightarrow M \models_X \phi.$$

Tuple Existence Logic and Independence Logic

Independence atoms in $\mathcal{T}$

$\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ is equivalent to

$$\forall \vec{p}_1\vec{p}_2\vec{p}_3(\neg(\vec{p}_1\vec{p}_2 \text{ @ } \vec{t}_1\vec{t}_2) \vee_{\vec{p}_1\vec{p}_2\vec{p}_3} \neg(\vec{p}_1\vec{p}_3 \text{ @ } \vec{t}_1\vec{t}_3) \vee_{\vec{p}_1\vec{p}_2\vec{p}_3}$$
$$\vee_{\vec{p}_1\vec{p}_2\vec{p}_3} \vec{p}_1\vec{p}_2\vec{p}_3 \text{ @ } \vec{t}_1\vec{t}_2\vec{t}_3).$$

Tuple Existence Atoms in $\mathcal{I}$

$\vec{t}_1 \text{ @ } \vec{t}_2$ is equivalent to

$$\forall u_1 u_2 \vec{z}((\vec{z} \neq \vec{t}_1 \wedge \vec{z} \neq \vec{t}_2) \vee (u_1 \neq u_2 \wedge \vec{z} \neq \vec{t}_2) \vee$$
$$\vee ((u_1 = u_2 \vee \vec{z} = \vec{t}_2) \wedge \vec{z} \perp_{\emptyset} u_1 u_2)).$$

Tuple Existence Logic and Independence Logic

Independence Logic is Tuple Existence Logic

- For every formula $\phi \in \mathcal{I}$ there exists a $\psi \in \mathcal{T}$ such that

$$M \models_X \phi \Leftrightarrow M \models_X \psi;$$

- For every formula $\psi \in \mathcal{T}$ there exists a $\phi \in \mathcal{I}$ such that

$$M \models_X \psi \Leftrightarrow M \models_X \phi.$$

Definability in Tuple Existence Logic

From Tuple Existence Logic to Σ_1^1

For every formula $\phi \in \mathcal{T}$ there exists a sentence $\phi' \in \Sigma_1^1$ such that $M \models_X \phi$ if and only if $M, \mathrm{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

From Σ_1^1 to Tuple Existence Logic

For every sentence $\phi'(R) \in \Sigma_1^1$ there exists a formula $\phi \in \mathcal{I}$ such that $M \models_X \phi$ if and only if $M, \mathrm{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Thanks to Juha Kontinen for pointing out this requirement!

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For every sentence $\phi'(R) \in \Sigma_1^1$ there exists a formula $\phi \in \mathcal{I}$ such that $M \models_X \phi$ if and only if $M, \mathrm{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Thanks to Juha Kontinen for pointing out this requirement!

Corollary: Definability on Independence Logic

From Independence Logic to Σ_1^1

For every formula $\phi \in \mathcal{I}$ there exists a sentence $\phi' \in \Sigma_1^1$ such that $M \models_X \phi$ if and only if $M, \text{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

From Σ_1^1 to Independence Logic

For every sentence $\phi'(R) \in \Sigma_1^1$ there exists a formula $\phi \in \mathcal{I}$ such that $M \models_X \phi$ if and only if $M, \text{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Left to Right

From Tuple Existence Logic to Σ_1^1

For every formula $\phi \in \mathcal{T}$ there exists a sentence $\phi' \in \Sigma_1^1$ such that $M \models_X \phi$ if and only if $M, \text{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Proof.

By structural induction over ϕ.

- If ϕ is a first order literal,

$$M \models_X \phi \Leftrightarrow M, \text{Rel}(X) \models \forall \vec{x}(R\vec{x} \rightarrow \phi);$$

Left to Right

From Tuple Existence Logic to Σ_1^1

For every formula $\phi \in \mathcal{T}$ there exists a sentence $\phi' \in \Sigma_1^1$ such that $M \models_X \phi$ if and only if $M, \mathrm{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Proof.

By structural induction over ϕ.

- If ϕ is a first order literal,

$$M \models_X \phi \Leftrightarrow M, \mathrm{Rel}(X) \models \forall \vec{x}(R\vec{x} \rightarrow \phi);$$

□

Proof (Continued).

- If ϕ is an inclusion dependency $\vec{t}_1 \subseteq \vec{t}_2$, $M \models_X \phi$ iff

$$M, \mathrm{Rel}(X) \models \forall \vec{x}_1 (R\vec{x}_1 \rightarrow \exists \vec{x}_2 (R\vec{x}_2 \wedge \vec{t}_1 \langle \vec{x}_1 \rangle = \vec{t}_2 \langle \vec{x}_2 \rangle));$$

- If ϕ is $\psi_1 \vee \psi_2$, let $\psi_1^*(R)$ and $\psi_2^*(R)$ be the corresponding Σ_1^1 sentences. Then

$$M \models_X \phi \Leftrightarrow M, \mathrm{Rel}(X) \models \exists Y \exists Z (Y \cup Z = R \wedge Y \cap Z = \emptyset \wedge \\ \wedge \psi_1^*(Y) \wedge \psi_2^*(Z)).$$

□

Proof (Finished).

- If ϕ is $\psi_1 \wedge \psi_2$, let $\psi_1^*(R)$ and $\psi_2^*(R)$ be the corresponding Σ_1^1 sentences. Then

$$M \models_X \phi \Leftrightarrow M, \mathrm{Rel}(X) \models \psi_1^*(R) \wedge \psi_2^*(R)) :$$

- If ϕ is $\exists x_{n+1}\psi$, $M \models_X \phi$ if and only if

$$M, \mathrm{Rel}(X) \models \exists Z(\forall \vec{x}(R\vec{x} \rightarrow \exists^{=1} x_{n+1}\, Z\vec{x}x_{n+1}) \wedge \psi^*(Z));$$

- If ϕ is $\forall x_{n+1}\psi$, $M \models_X \phi$ if and only if

$$M, \mathrm{Rel}(X) \models \exists Z(\forall \vec{x}(R\vec{x} \rightarrow \forall x_{n+1}\, Z\vec{x}x_{n+1}) \wedge \psi^*(Z)).$$

□

Right to Left

From Σ_1^1 to Tuple Existence Logic

For every sentence $\phi'(R) \in \Sigma_1^1$ there exists a formula $\phi \in \mathcal{I}$ such that $M \models_X \phi$ if and only if $M, \mathrm{Rel}(X) \models \phi'$ for all suitable M and all **nonempty** X.

Proof.

Similar to the ones in KV 2009 and KN 2009.
Write $\phi'(R)$ as $\exists R' \exists \vec{f}\, \forall \vec{z}((R'\vec{x} \leftrightarrow R\vec{x}) \wedge \psi(R', \vec{z}))$ where $\vec{x}$ subsequence of $\vec{z}$,
ψ quantifier free, R not in ψ, each f_i only as $f_i(\vec{w}_i)$ for some fixed $\vec{w}_i \subseteq \vec{z}$, R' only as $R'\vec{x}$. □

Right to Left

Proof (continued).

Write $\phi'(R)$ as $\exists R' \exists \vec{f}\, \forall \vec{z}((R'\vec{x} \leftrightarrow R\vec{x}) \wedge \psi(R', \vec{z}))$ where $\vec{x}$ subsequence of $\vec{z}$,
ψ quantifier free, R not in ψ, each f_i only as $f_i(\vec{w}_i)$ for some fixed $\vec{w}_i \subseteq \vec{z}$, R' only as $R'\vec{x}$.
Then $M, \mathrm{Rel}(X) \models \phi'$ if and only if

$$M, \mathrm{Rel}(X) \models \exists g_1 g_2 \exists \vec{f}\, \forall \vec{z}((f_1(\vec{x}) = f_2(\vec{x}) \leftrightarrow R\vec{x}) \wedge \psi'(\vec{z}))$$

where $\psi' = \psi[f_1\vec{x} = f_2\vec{x}/R\vec{x}]$. □

Right to Left

Proof (continued).

$$\phi' \equiv \exists g_1 g_2 \exists \vec{f}\, \forall \vec{z}((g_1(\vec{x}) = g_2(\vec{x}) \leftrightarrow R\vec{x}) \wedge \psi'(\vec{z}))$$

where $\psi' = \psi[g_1\vec{x} = g_2\vec{x}/R\vec{x}]$.

Then, if X nonempty, $\mathrm{Dom}(X) = \vec{y}$, $M, \mathrm{Rel}(X) \models \phi'$ iff

$$M \models_X \forall \vec{z} \exists u_1 u_2 \vec{v} \left(\left(\bigwedge_{i=1}^{2} {=}(\vec{x}, u_i) \wedge \bigwedge_{j} {=}(\vec{w}_j, v_j) \right) \wedge \right.$$
$$\left. \wedge\, ((\vec{x} \circledcirc \vec{y} \wedge u_1 = u_2) \vee (\neg \vec{x} \circledcirc \vec{y} \wedge u_1 \neq u_2)) \wedge \theta \right)$$

where θ is $\psi'[u_1/g_1\vec{x}][u_2/g_2\vec{x}][\vec{w}/\vec{f}\vec{w}]$. □

Right to Left

Proof (continued).

Suppose that, for all s with domain $\vec{z}$,

$$M, \mathrm{Rel}(X), g_1, g_2, \vec{f} \models (g_1(\vec{x}) = g_2(\vec{x}) \leftrightarrow R\vec{x}) \wedge \psi'(\vec{z}).$$

Extend X to Y choosing the u_1, u_2, $\vec{v}$ according to g_1, g_2, $\vec{f}$.

- $M \models_Y \bigwedge_{i=1}^{2} =\!(\vec{x}, u_i) \wedge \bigwedge_j =\!(\vec{w}_j, v_j)$: obvious;
- $M \models_Y \theta$: by construction;
- $M \models_Y (\vec{x} \text{ @ } \vec{y} \wedge u_1 = u_2) \vee (\neg\vec{x} \text{ @ } \vec{y} \wedge u_1 \neq u_2)$:
  If $u_1 = u_2$, $\vec{x} \in \mathrm{Rel}(X)$, so $\vec{x} \text{ @ } \vec{y}$;
  If $u_1 \neq u_2$, $\vec{x} \notin \mathrm{Rel}(X)$, so $\neg\vec{x} \text{ @ } \vec{y}$.

□

Right to Left

Proof (continued).

Conv., suppose X nonempty, $Y = X[M/\vec{z}][G_1/u_1][G_2/u_2][\vec{F}/\vec{v}]$,

$$M \models_Y \bigwedge_{i=1}^{2} =\!(\vec{x}, u_i) \wedge \bigwedge_j =\!(\vec{w}_j, v_j),$$
$$M \models_Y (\vec{x} \, @ \, \vec{y} \wedge u_1 = u_2) \vee (\neg \vec{x} \, @ \, \vec{y} \wedge u_1 \neq u_2),$$
$$M \models_Y \theta.$$

Choose $g_1(\vec{x}), g_2(\vec{x}), \vec{f}(\vec{w})$ according to $G_1, G_2, \vec{F}$.
Let s be any assignment, domain = $\vec{z}$. □

Right to Left

Proof (continued).

Choose $g_1(\vec{x}), g_2(\vec{x}), \vec{f}(\vec{w})$ according to $G_1, G_2, \vec{F}$.
Let s be any assignment, domain = $\vec{z}$.

- $M, \mathrm{Rel}(R), g_1, g_2, \vec{f} \models_s \psi'$: Take $h \in X$. Then $h[s/\vec{z}][g_1/u_1][g_2/u_2][\vec{f}/\vec{v}] \in Y$, $M \models_Y \theta$.
- $M, \mathrm{Rel}(R), g_1, g_2, \vec{f} \models_s g_1(\vec{x}) = g_2(\vec{x}) \leftrightarrow R\vec{x}$:
  Suppose $g_1(\vec{x}) = g_2(\vec{x})$, let $h \in X$.
  Consider $o = h[s/\vec{z}][g_1/u_1][g_2/u_2][\vec{f}/\vec{v}]$:
  $o \in Y_1, M \models_{Y_1} \vec{x} @ \vec{y}$. So $\exists o' \in Y_1$, $o'(\vec{y}) = o(\vec{x})$, so $s(\vec{x}) = o(\vec{x}) \in \mathrm{Rel}(X)$.

□

Right to Left

Proof (finished).

Choose $g_1(\vec{x}), g_2(\vec{x}), \vec{f}(\vec{w})$ according to $G_1, G_2, \vec{F}$.
Let s be any assignment, domain = $\vec{z}$.

- $M, \text{Rel}(R), g_1, g_2, \vec{f} \models_s g_1(\vec{x}) = g_2(\vec{x}) \leftrightarrow R\vec{x}$:
  Suppose $g_1(\vec{x}) \neq g_2(\vec{x})$, let $h \in X$.
  Consider $o = h[s/\vec{z}][g_1/u_1][g_2/u_2][\vec{f}/\vec{v}]$:
  $o \in Y_2, M \models_{Y_2} \neg\vec{x} \circledcirc \vec{y}$. So $\forall o' \in Y_2$, $o'(\vec{y}) \neq o(\vec{x})$.
  But for all $h' \in X$, $o' = h'[s/\vec{z}][g_1/u_1][g_2/u_2][\vec{f}/\vec{v}] \in Y_2$;
  then, for all such h',
  $s(\vec{x}) = o(\vec{x}) \neq o'(\vec{y}) = h'(\vec{y})$.
  Therefore, $s(\vec{x}) \not\in \text{Rel}(X)$.

□

Definability in Tuple Existence Logic

Equality Generating Dependencies

$$\forall \vec{x}(R\vec{t}_1 \wedge \ldots \wedge R\vec{t}_n \rightarrow t_{n+1} = t_{n+2})$$

Tuple Generating Dependencies

$$\forall \vec{x}(R\vec{t}_1 \wedge \ldots \wedge R\vec{t}_n \rightarrow \exists \vec{y} R\vec{t}')$$

Corollary

All Tuple Generating and Equality Generating Dependencies are expressible in Independence Logic (or in $\mathcal{T}$).

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Outline

Two Semantics for Disjuction

A lax semantics

$$M \models_X \psi_1 \vee^L \psi_2 \Leftrightarrow \exists Y, Z \text{ s.t. } X = Y \cup Z, M \models_Y \psi_1 \text{ and } M \models_Z \psi_2;$$

A strict semantics

$$M \models_X \psi_1 \vee^S \psi_2 \Leftrightarrow \exists Y, Z \text{ s.t. } X = Y \cup Z, X \cap Y = \emptyset, M \models_Y \psi_1 \text{ and } M \models_Z \psi_2;$$

$\mathcal{D}$ is usually given with $\exists^L$ (or even: $X \subseteq Y \cup Z$!).

Two Semantics for Disjuction

A lax semantics

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$\mathcal{D}$ is usually given with $\exists^L$ (or even: $X \subseteq Y \cup Z$!).

In Dependence Logic, Lax $=$ Strict

No difference for $\mathcal{D}$ (or for $\mathcal{T}^-$)

If $\psi_1, \psi_2 \in \mathcal{D}$, $M \models_X \psi_1 \vee^S \psi_2$ iff $M \models_X \psi_1 \vee^L \psi_2$.

Proof.

- If $M \models_X \psi_1 \vee^S \psi_2$, $M \models_X \psi_1 \vee^L \psi_2$;
- If $M \models_X \psi_1 \vee^L \psi_2$ then $X = X_1 \cup X_2$, $M \models_{X_1} \psi_1$, $M \models_{X_2} \psi_2$. Take $Y = X_2 \backslash X_1$: **by downwards closure**, $M \models_Y \psi_2$, $X_1 \cup Y = X$, so $M \models_X \psi_1 \vee^S \psi_2$.

□

In Inclusion or Independence Logic, Lax $\neq$ Strict

Different for $\mathcal{T}^+$ (and for $\mathcal{T}$, and for $\mathcal{I}$)!

There exist M, X and $\psi_1, \psi_2 \in \mathcal{T}^+$ such that

$$M \models_X \psi_1 \vee^L \psi_2 \text{ but } M \not\models_X \psi_1 \vee^S \psi_2.$$

Proof.

Let $X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

and $\mathrm{Dom}(M) = \{0 \ldots 4\}$. Then

$$M \models_X (x \text{ @ } y) \vee^L (y \text{ @ } z), \; M \not\models_X (x \text{ @ } y) \vee^S (y \text{ @ } z).$$

□

In Inclusion or Independence Logic, Lax $\neq$ Strict

Proof (continued).

$X =$

	x	y	z
s_0	0	1	2
s_1	1	0	3
s_2	4	3	0

- $M \models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $Y = \{s_0, s_1\}, Z = \{s_1, s_2\}$.
  $M \models_Y x \subseteq y, \quad M \models_Z y \subseteq z$.

In Inclusion or Independence Logic, Lax $\neq$ Strict

Proof (continued).

$X =$

	x	y	z
s_0	0	1	2
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- $M \models_X (x \mathbin{@} y) \vee^L (y \mathbin{@} z)$:
  Let $Y = \{s_0, s_1\}, Z = \{s_1, s_2\}$.
  $M \models_Y x \mathbin{@} y$, $M \models_Z y \mathbin{@} z$.

□

In Inclusion or Independence Logic, Lax $\neq$ Strict

Proof (continued).

$X =$

	x	y	z
s_0	0	1	2
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- $M \models_X (x \,@\, y) \vee^L (y \,@\, z)$:
  Let $Y = \{s_0, s_1\}, Z = \{s_1, s_2\}$.
  $M \models_Y x \,@\, y, \quad M \models_Z y \,@\, z$.

□

In Inclusion or Independence Logic, Lax $\neq$ Strict

Proof (finished).

$X =$

	x	y	z
s_0	0	1	2
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- $M \not\models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $X = Y \cup Z$, $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.
  $s_2 \notin Y$, so $s_2 \in Z$, so $s_1 \in Z$;
  $s_0 \notin Z$, so $s_0 \in Y$, so $s_1 \in Y$.
  So $Y \cap Z \neq \emptyset$.

In Inclusion or Independence Logic, Lax $\neq$ Strict

Proof (finished).

$X =$

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  Let $X = Y \cup Z$, $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.
  $s_2 \notin Y$, so $s_2 \in Z$, so $s_1 \in Z$;
  $s_0 \notin Z$, so $s_0 \in Y$, so $s_1 \in Y$.
  So $Y \cap Z \neq \emptyset$.

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  Let $X = Y \cup Z$, $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.
  $s_2 \notin Y$, so $s_2 \in Z$, so $s_1 \in Z$;
  $s_0 \notin Z$, so $s_0 \in Y$, so $s_1 \in Y$.
  So $Y \cap Z \neq \emptyset$.

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- $M \not\models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $X = Y \cup Z$, $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.
  $s_2 \not\in Y$, so $s_2 \in Z$, so $s_1 \in Z$;
  $s_0 \not\in Z$, so $s_0 \in Y$, so $s_1 \in Y$.
  So $Y \cap Z \neq \emptyset$.

In Inclusion or Independence Logic, Lax $\neq$ Strict

Proof (finished).

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s_0	0	1	2
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- $M \not\models_X (x \subseteq y) \vee^L (y \subseteq z)$:
  Let $X = Y \cup Z$, $M \models_Y x \subseteq y$, $M \models_Z y \subseteq z$.
  $s_2 \notin Y$, so $s_2 \in Z$, so $s_1 \in Z$;
  $s_0 \notin Z$, so $s_0 \in Y$, so $s_1 \in Y$.
  So $Y \cap Z \neq \emptyset$.

$\square$

From Strict to Lax Disjunction

From strict to lax

If z not in ψ_1, ψ_2,

$$M \models_X \psi_1 \vee^L \psi_2 \Leftrightarrow M \models_X \forall z(\psi_1 \vee^S \psi_2).$$

Proof.

Let $0 \in \mathrm{Dom}(M)$, assume $|\mathrm{Dom}(M)| \geq 2$.
Suppose $X = Y \cup Z$, $M \models_Y \psi_1$, $M \models_Z \psi_2$, and let $W = Y \cap Z$.
Now define

$$Y' = (Y \backslash W)[M/z] \cup (W[0/z]), Z' = Z[M/z] \backslash Y'.$$

Then $Y' \cap Z' = \emptyset$, $Y' \cup Z' = X[M/z]$, $M \models_{Y'} \psi_1$, $M \models_{Z'} \psi_2$. □

Trivial Quantification and $\vee^S$

Corollary: $\vee^S$ is not invariant under trivial quantifications!

There exist formulas ψ_1 and $\psi_2 \in \mathcal{T}$, such that z does not occur in ψ_1, ψ_2 but

$$\psi_1 \vee^S \psi_2 \not\equiv \forall z(\psi_1 \vee^S \psi_2).$$

Trivial Quantification and $\vee^L$

$\vee^L$ invariant under trivial quantification

For all ψ_1 and ψ_2 in $\mathcal{T}$ and all $z \notin \psi_1, \psi_2$,

$$\psi_1 \vee^S \psi_2 \not\equiv \forall z(\psi_1 \vee^S \psi_2).$$

Proof.

Obvious from definition: if $X = Y \cup Z$, $M \models_Y \psi_1$, $M \models_Y \psi_2$, then $X[M/z] = Y[M/z] \cup Z[M/z]$, $M \models_{Y[M/z]} \psi_1$, $M \models_{Z[M/z]} \psi_2$. □

This strongly suggests that we want $\vee^L$ in our semantics.

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Outline

Two Semantics for Existentials

A strict semantics

$$M \models_X \exists^S x\psi \Leftrightarrow \exists F : X \to M \text{ s.t. } M \models_{X[F/x]} \psi,$$

for $X[F/x] = \{s[F(s)/x] : s \in X\}$;

A lax semantics

$$M \models_X \exists^L x\psi \Leftrightarrow \exists F : H \to \mathcal{P}(M)\backslash\{\emptyset\} \text{ s.t. } M \models_{X[F/x]} \psi,$$

for $X[H/x] = \{s[m/x] : s \in X, m \in H(s)\}$.

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$\mathcal{D}$ is usually given with $\exists^S$.

In Dependence Logic, Strict = Lax

No difference for $\mathcal{D}$ (or for $\mathcal{T}^-$)

If $\psi \in \mathcal{D}$, $M \models_X \exists^S x \psi$ iff $M \models_X \exists^L x \psi$ (using AC).

Proof.

- If $M \models_X \exists^S x \psi$, $M \models_X \exists^L x \psi$;
- If $M \models_X \exists^L x \psi$, $M \models_{X[H/x]} \psi$ for some $H : X \to \mathcal{P}(M) \backslash \{\emptyset\}$. Let $F : X \to M$ be such that $F(s) \in H(s)$ for all $s \in X$: then $X[F/x] \subseteq X[H/x]$, so by downward closure $M \models_{X[F/x]} \psi$. Then $M \models_X \exists^S x \psi$, as required.

□

In Independence and Inclusion Logic, Strict $\neq$ Lax

Different for $\mathcal{T}^+$ (and for $\mathcal{T}$, and for $\mathcal{I}$)!

There exist M, X and $\psi \in \mathcal{T}^+$ such that

$$M \models_X \exists^L x \psi \text{ but } M \not\models_X \exists^S \psi.$$

Proof.

Let $\text{Dom}(M) = \{0, 1, 2\}$, $P^M = \{(0, 2), (1, 0), (1, 1)\}$, and $X = \{s_0, s_1\}$ for $s_0 = (y : 0)$, $s_1 = (y : 1)$.
Then
$M \models_X \exists^L x(y \subseteq x \wedge Pyx)$ but $M \not\models_X \exists^S x(y \subseteq x \wedge Pyx)$. $\square$

In Independence and Inclusion Logic, Strict $\neq$ Lax

Proof (continued).

$\text{Dom}(M) = \{0, 1, 2\}$, $P^M = \{(0, 2), (1, 0), (1, 1)\}$, and $X = \{s_0, s_1\}$ for $s_0 = (y : 0)$, $s_1 = (y : 1)$.

- $M \models_X \exists^L x(y \subseteq x \wedge Pyx)$: let $H : X \to \mathcal{P}(M)$ be such that $H(s_0) = \{2\}$, $H(s_1) = \{0, 1\}$. Then

$X[H/x] =$

	y	x
s'_0	0	2
s'_1	1	0
s'_2	1	1.

and this team satisfies $y \subseteq x$ and Pyx.

□

In Independence and Inclusion Logic, Strict $\neq$ Lax

Proof (finished).

$\mathrm{Dom}(M) = \{0, 1, 2\}$, $P^M = \{(0, 2), (1, 0), (1, 1)\}$, and $X = \{s_0, s_1\}$ for $s_0 = (y : 0)$, $s_1 = (y : 1)$.

- $M \not\models_X \exists^S x(y \subseteq x \wedge Pyx)$: take any $F : X \to M$, and consider $X[F/x]$.
  If $F(s_0) \neq 2$, $M \not\models_{X[F/x]} Pyx$; so $F(s_0) = 2$.
  But then

$$X[F/x] = \begin{array}{c|c|c} & y & x \\ \hline s'_0 & 0 & 2 \\ s'_1 & 1 & F(s_1) \end{array}$$

  and $M \not\models_{X[F/x]} y \subseteq x$, since $F(s_1) \neq 0$ or $F(s_1) \neq 1$.

$\square$

From Strict to Lax Existentials

From strict to lax semantics

If z not in ψ and $z \neq x$,

$$M \models_X \exists^L x \psi \Leftrightarrow M \models_X \forall z \exists^L x \psi.$$

From Strict to Lax Existentials

Proof.

Suppose that for $H : X \to \mathcal{P}(X) \backslash \{\emptyset\}$, $M \models_{X[H/x]} \psi$.
For every $s \in X$, let $m_s \in H(s)$; then define $F : X[M/z] \to M$ as

$$F(s[m/z]) = \begin{cases} m & \text{if } m \in H(s); \\ m_s & \text{otherwise.} \end{cases}$$

Forgetting the variable z, $X[M/z][F/x]$ is precisely $X[H/z]$; hence,
$M \models_{X[M/z][F/x]} \psi$, as required (other direction is trivial). □

Trivial Quantification and $\exists^S$

Corollary: $\exists^S$ is not invariant under trivial quantifications!

There exists a $\psi \in \mathcal{T}$, such that z does not occur in it but

$$\exists^S x \psi \not\equiv \forall z \exists^S x \psi.$$

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$\exists^L$ invariant under trivial quantification

For all ψ in $\mathcal{T}$ and all $z \notin \psi$,

$$\exists^L x \psi \equiv \forall z \exists^L s \psi.$$

Proof.

If for $H : X[M/z] \to \mathcal{P}(M)$ it holds that $M \models_{X[M/z][H/x]} \psi$, define $H' : X \to \mathcal{P}(M)$ as

$$H'(s) = \{m \in M : \exists m' \in M \text{ s.t. } m \in H(s[m'/z])\}.$$

Then $M \models_{X[H'/z]} \psi$, as required. □

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Outline

Strict Existentials on a Fixed Domain

Recovering strict existentials

Let $\vec{x}$ be a fixed tuple of variables. Then, for all teams X with $\mathrm{Dom}(X) = \vec{x}$, for all z and all $\psi \in \mathcal{T}$,

$$M \models_X \exists^S z \psi \Leftrightarrow M \models_X \exists^L z (=(\vec{x}, z) \wedge \psi).$$

Recovering strict disjunctions

Let $\vec{x}$ be a fixed tuple of variables. Then, for all teams X with $\mathrm{Dom}(X) = \vec{x}$, for all z and all $\psi_1, \psi_2 \in \mathcal{T}$,

$$M \models_X \psi_1 \vee^S \psi_2 \Leftrightarrow M \models_X \exists^S z_1 z_2 ((z_1 = z_2 \wedge \psi_1) \vee^L (z_1 \neq z_2 \wedge \psi_2)).$$

Recovering Strict Operators in $\mathcal{T}$ and $\mathcal{I}$

Corollary

As long as the domain of the team X is fixed, we can use the strict semantics for $\mathcal{T}$ or $\mathcal{I}$, and the result will be transferable to the lax one.

This does not necessarily hold for Inclusion Logic $\mathcal{T}^+$!

Convention: $\mathcal{T}^+$ has the lax semantics, unless otherwise specified.

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Another question

What we know so far

$$FOL \subseteq \mathcal{T}^- \equiv \mathcal{D} \subseteq \mathcal{I} \equiv \mathcal{T};$$
$$FOL \subseteq \mathcal{T}^+ \subseteq \mathcal{I} \equiv \mathcal{T}.$$

What about $\mathcal{T}^+$?

- Is it stronger than First Order Logic?
- Is it contained in Dependence Logic?
- Does it contain Dependence Logic?
- Is it Independence Logic and $\mathcal{T}$ already?

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$\mathcal{T}^+$ is not downward closed

A simple Inclusion Logic formula

$M \models_X \forall y (y \subseteq x)$ if and only if $X(x) \in \{\emptyset, M\}$.

Proof.

If $X \neq \emptyset$,

$$M \models_X \forall y (y \subseteq x) \Leftrightarrow M \models_{X[M/y]} y \subseteq x \Leftrightarrow$$
$$\Leftrightarrow \forall m \in M \exists s \in X \text{ s.t. } s(x) = m \Leftrightarrow X(x) = M.$$

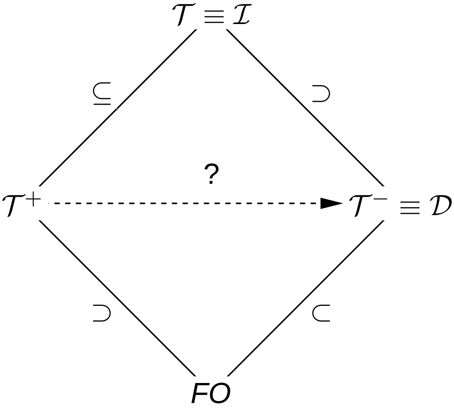
□

$\mathcal{T}^+$ is not downward closed

Corollary

- First Order Logic is properly contained in $\mathcal{T}^+$;
- Dependence Logic does not contain $\mathcal{T}^+$.

Inclusions between logics of imperfect information



A Closure Property for $\mathcal{T}^+$

Theorem

Let $\phi \in \mathcal{T}^+$. Then, for all M and all X, Y,

$$M \models_X \phi,\ M \models_Y \phi \Rightarrow M \models_{X \cup Y} \phi.$$

Proof.

By induction over ϕ.

- If ϕ is a first order literal, obvious;
- If ϕ is $\vec{t}_1 \subseteq \vec{t}_2$, take $s \in X \cup Y$.
  - If $s \in X$, since $M \models_X \vec{t}_1 \subseteq \vec{t}_2$ there is $s' \in X$, $\vec{t}_1\langle s \rangle = \vec{t}_2\langle s' \rangle$;
  - If $s \in Y$, since $M \models_Y \vec{t}_1 \subseteq \vec{t}_2$ there is $s' \in Y$, $\vec{t}_1\langle s \rangle = \vec{t}_2\langle s' \rangle$.

  So $\exists s' \in X \cup Y$, $\vec{t}_1\langle s \rangle = \vec{t}_2\langle s' \rangle$, and $M \models_{X \cup Y} \vec{t}_1 \subseteq \vec{t}_2$;

□

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  - If $s \in Y$, since $M \models_Y \vec{t}_1 \subseteq \vec{t}_2$ there is $s' \in Y$, $\vec{t}_1\langle s\rangle = \vec{t}_2\langle s'\rangle$.

  So $\exists s' \in X \cup Y$, $\vec{t}_1\langle s\rangle = \vec{t}_2\langle s'\rangle$, and $M \models_{X \cup Y} \vec{t}_1 \subseteq \vec{t}_2$;

$\square$

A Closure Property for $\mathcal{T}^+$

Proof (continued):

- If $M \models_X \psi_1 \vee \psi_2$ and $M \models_Y \psi_1 \vee \psi_2$, then $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$ and

$$M \models_{X_i} \psi_i, \; M \models_{Y_i} \psi_i \text{ for } i = 1, 2.$$

  So $M \models_{X_1 \cup Y_1} \psi_1$, $M \models_{X_2 \cup Y_2} \psi_2$ and $M \models_{X \cup Y} \psi_1 \vee \psi_2$.
- If $M \models_X \psi_1 \wedge \psi_2$ and $M \models_Y \psi_1 \wedge \psi_2$ then

$$M \models_X \psi_1, M \models_X \psi_2, M \models_Y \psi_1, M \models_Y \psi_2.$$

  Then $M \models_{X \cup Y} \psi_1$, $M \models_{X \cup Y} \psi_2$ and $M \models_{X \cup Y} \psi_1 \wedge \psi_2$.

□

A Closure Property for $\mathcal{T}^+$

Proof (finished):

- If $M \models_X \exists z \phi$ and $M \models_Y \exists z \psi$, then $M \models_{X[H_1/z]} \psi$ and $M \models_{Y[H_2/z]} \psi$ for some F, G. By induction hypothesis, $M \models_{X[H_1/z] \cup H_2[G/z]} \psi$. Now let $H = H_1 \cup H_2$: then $(X \cup Y)[H/z] = X[H_1/z] \cup Y[H_2/z]$ and hence $M \models_{X \cup Y} \exists z \psi$.
- If $M \models_X \forall z \psi$ and $M \models_Y \forall z \psi$, $M \models_{X[M/z]} \psi$ and $M \models_{X[M/z]} \psi$. But $(X \cup Y)[M/z] = X[M/z] \cup Y[M/z]$ and $M \models_{X[M/z] \cup Y[M/z]} \psi$ by induction hypothesis, so $M \models_{X \cup Y} \forall z \psi$.

□

A closure property for $\mathcal{T}^+$

Theorem

Let $\phi \in \mathcal{T}^+$ (Strict Semantics). Then, for all M and all X, Y with $X \cap Y = \emptyset$,

$$M \models_X \phi,\ M \models_Y \phi \Rightarrow M \models_{X \cup Y} \phi.$$

Proof.

Just like the previous one. □

Dependence Logic and $\mathcal{T}^+$

Corollary: $\mathcal{D} \not\subseteq \mathcal{T}^+$

Dependence Logic is not contained in $\mathcal{T}^+$, and

$$FOL \subsetneq T^+ \subsetneq \mathcal{T} \equiv \mathcal{I}.$$

Proof of the corollary:

Consider $X = \{(x:0)\}$, $Y = \{(x:1)\}$.
Then $M \models_X =\!(x)$ and $M \models_Y =\!(x)$, but

$$M \not\models_{X \cup Y} =\!(x).$$

Hence, constancy atoms are not expressible in Inclusion Logic. □

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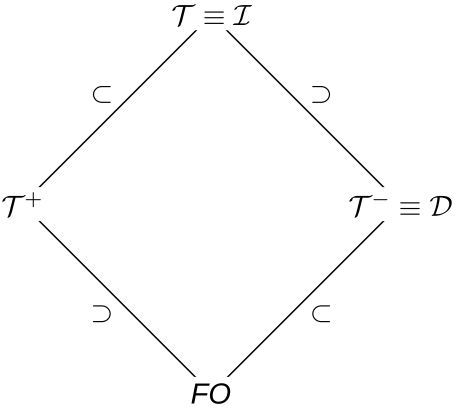

$\mathcal{T}^+$ and T^- for Quantifier Free Formulas

Corollary

Let $\phi \in \mathcal{T}$ be quantifier-free. If ϕ is expressible in Dependence Logic and in Inclusion Logic then it is expr. in *FO*.

Proof of the corollary:

ϕ expr. in Dependence Logic, so ϕ is downwards closed:

$$M \models_X \phi \Rightarrow \forall s \in X, M \models_{\{s\}} \phi.$$

ϕ expr. in Inclusion Logic, so ϕ is closed under unions:

$$\forall s \in X, M \models_{\{s\}} \phi \Rightarrow M \models_X \phi.$$

Therefore ϕ is *flat*, and, by (Väänänen 2007), it is first order. □

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A Conjecture: $\mathcal{T}^+ \cap \mathcal{T}^- \equiv FO$

Conjecture

Let $\phi \in \mathcal{T}^+$, $\phi' \in \mathcal{T}^-$ (or $\phi' \in \mathcal{D}$) be such that

$$M \models_X \phi \Leftrightarrow M \models_X \phi'$$

for all suitable M and X. Then there exists a $\phi'' \in FO$ s.t.

$$\phi \equiv \phi' \equiv \phi''.$$

Perhaps prove using Craig Interpolation (as in KV 2009)?

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