From Dependence to Independence

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LogICCC project "Logic of Interaction" (LINT)

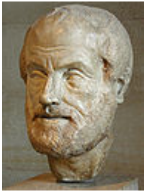
- Partners: Aachen, Amsterdam, Gothenburg, Tampere - Helsinki
- Associated: Oxford, Paris
- Joint work with Erich Grädel (Aachen)

1. Introduction

- Probability and statistics: Random variables
- Mathematics: Equations, linear dependence, algebraic dependence
- Philosophy: Causality
- Computer science: Data mining
- Logic: "Logic of Interaction" (LINT)

Now some examples:

2. Examples

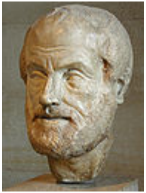


Balls of identical size but different weights
are dropped from different heights.

Aristotle: The heavier the ball, the shorter the time of descent.

Height (m)	Weight (kg)	Time (s)
20	1.0	2.0
20	1.2	2.0
20	1.4	2.0
30	1.0	2.5
30	1.2	2.5
30	1.4	2.5
40	1.0	2.8
40	1.2	2.8
40	1.4	2.8

We can think of this table as a set of assignments of values to three variables: h, w and t.

Balls of identical size but different weights
are dropped from different heights.

Galileo: The time t of descent is completely determined by the height h but completely independent of the weight w.
First order logic: $t \cdot t \cdot g = 2 \cdot h$ and no variable w occurs here.

2. Examples

Aristotle: The sex of the offspring is determined by species, the environment and the nutrients.

2. Examples

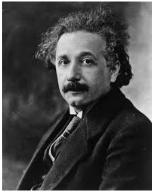
Species	Sex chromosomes	Sex
human	XY	male
human	XX	female
horse	XY	male
horse	XX	female
fruit fly	XY	male
fruit fly	XX	female

We can think of this table as a set of assignments of values to three variables: *species*, *sex_chromosomes* and *sex*.

2. Examples

C. E. McClung 1902: Sex is completely determined by the XY-chromosomes, independently of the species, environment and the nutrients.

2. Examples



- The speed of light in vacuum, measured by a non-accelerating observer, is independent of the motion of the observer or the source.
- Sun rises every morning independently of whether I rise from my bed or not.
- $2 + 2 = 4$ independently of anything.
- **Lesson:** Being a constant is a form of independence.

- When can we say that A depends on B?
- Perhaps, if A has a definition where B occurs.
- What if A has no definition, just a list of values?
- We now focus on the strongest form of dependence.

3. Dependence

a	b	$R(a,b)$
0	0	1
0	1	1
1	0	0
1	1	0

- We are told $a = 0$. Can we tell the truth-value of $R(a,b)$? Yes.
- We are told $a = 1$. Can we tell the truth-value of $R(a,b)$? Yes.
- a totally determines $R(a,b)$.

3. Dependence

x	y	z
0	0	1
0	1	1
1	0	0
1	1	0

- We are told $x = 0$. Can we tell the value of z? Yes.
- We are told $x = 1$. Can we tell the value of z? Yes.
- x totally determines z.

3. Dependence

x	y	z
0	0	1
0	1	1
1	0	0
1	1	0

- We are told the value of x. Can we tell the value of z? Yes.
- x totally determines z.

3. Dependence

x	y	z
0	0	10
3	1	100
7	0	0
12	1	10^5

- We are told the value of x. Can we tell the value of z? Yes.
- x totally determines z.
- We call this functional dependence.

Dependence atom

$$=(\vec{x}, y),$$

which is like a weak version of:

$$x = y.$$

If we adopt the shorthand

$$=(\vec{x}, \vec{y}) \text{ for } =(\vec{x}, y_1) \wedge \ldots \wedge =(\vec{x}, y_n)$$

we get a more general functional dependence. Although there are many different intuitive meanings for $=(\vec{x}, \vec{y})$, such as "$\vec{x}$ totally determines $\vec{y}$" or "$\vec{y}$ is a function of $\vec{x}$", the best way to understand the concept is to give it semantics:

3. Dependence

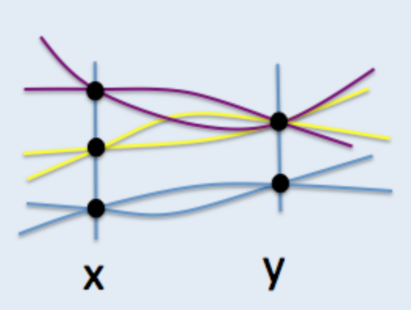
Definition

Sets of assignments are called *teams*. A team X satisfies $=(\vec{x}, \vec{y})$ if

$$\forall s, s' \in X(s(\vec{x}) = s'(\vec{x}) \to s(\vec{y}) = s'(\vec{y})).$$

This condition is a universal statement. As a consequence it is closed downward, that is, if a team satisfies it, every subteam does. In particular, the empty team satisfies it for trivial reasons. Also, every singleton team $\{s\}$ satisfies it, again for trivial reasons.

3. Dependence



3. Dependence

1. $=(\vec{x}, \vec{x})$.
2. If $=(\vec{y}, \vec{x})$ and $\vec{y} \subseteq \vec{z}$, then $=(\vec{z}, \vec{x})$.
3. If $\vec{y}$ is a permutation of $\vec{z}$, $\vec{u}$ is a permutation of $\vec{x}$, and $=(\vec{z}, \vec{x})$, then $=(\vec{y}, \vec{u})$.
4. If $=(\vec{y}, \vec{z})$ and $=(\vec{z}, \vec{x})$, then $=(\vec{y}, \vec{x})$.

3. Dependence

Theorem (Armstrong 1974)

The rules (1)-(4) completely describe $=(\vec{y}, \vec{x})$ *in the following sense: If* T *is a finite set of dependence atoms of the form* $=(\vec{y}, \vec{x})$ *for various* $\vec{x}$ *and* $\vec{y}$*, then* $=(\vec{y}, \vec{x})$ *follows from* T *according to the above rules if and only if every team that satisfies* T *also satisfies* $=(\vec{y}, \vec{x})$*.*

4. Independence

- When can we say that A is independent of B?
- Surely, if A has a definition where B does not occur at all.
- What if A has no definition, just a list of values?
- Again, we focus on the strongest conceivable form of independence, a kind of total independence (or "freeness") like we above focused on total dependence (or "determination").

4. Independence

a	b	c	$R(a,b,c)$
0	0		0
0	1		1
1	0		1
1	1		0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- We are told $R(a,b,c)$ is true. Can we tell what a is? No.
- We are told $R(a,b,c)$ is false. Can we tell what a is? No.
- We are told $a = 0$. Can we tell the truth-value of $R(a,b,c)$? No.
- We are told $a = 1$. Can we tell the truth-value of $R(a,b,c)$? No.
- $R(a,b,c)$ and a are totally independent of each other.

4. Independence

x	y	z	u
0	0		0
0	1		1
1	0		1
1	1		0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- We are told $u = 1$. Can we tell what x is? No.
- We are told $u = 0$. Can we tell what x is? No.
- We are told $x = 0$. Can we tell the value of u? No.
- We are told $x = 1$. Can we tell the value of u? No.
- u and x are totally independent of each other.

4. Independence

x	y	z	u
0	0		0
0	1		1
1	0		1
1	1		0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- We are told the value of u. Can we tell what x is? No.
- We are told the value of x. Can we tell the value of u? No.
- u and x are totally independent of each other.

4. Independence

x	y	z	u
5			10
13			18
5			18
13			10

- We are told the value of u. Can we tell what x is? No.
- We are told the value of x. Can we tell the value of u? No.
- u and x are totally independent of each other.

4. Independence

Definition

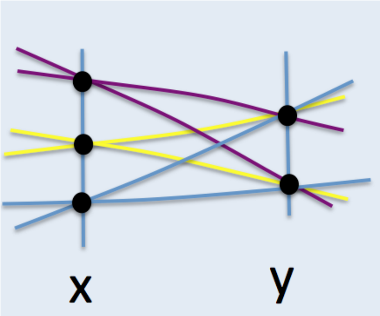
A team X satisfies the atomic formula $y \perp x$ if

$$\forall s, s' \in X \exists s'' \in X(s''(y) = s(y) \wedge s''(x) = s'(x)).$$

4. Independence

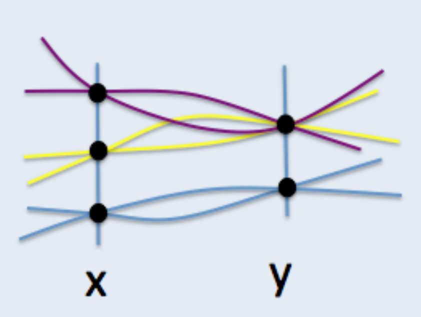
	y	u	z	x
s	5			10
s'	13			18
s''	5			18
	13			10

4. Independence

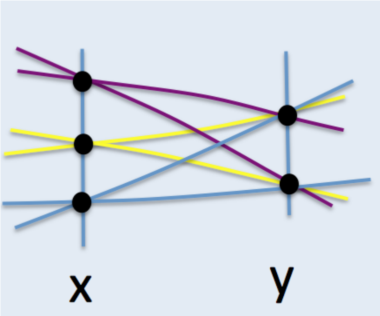


Independence

4. Independence



Dependence



Independence

4. Independence

Definition

The following rules are called the **Independence Axioms**

1. If $x \perp y$, then $y \perp x$ (Symmetry Rule).
2. If $x \perp x$, then $y \perp x$ (Constancy Rule).

For the Constancy Rule, remember that a constant value is independent of everything.

A new Armstrong Theorem:

Theorem (Completeness of the Independence Axioms)

If T is a finite set of dependence atoms of the form $u \perp v$ for various u and v, then $y \perp x$ follows from T according to the above rules if and only if every team that satisfies T also satisfies $y \perp x$.

5. Richer independence

Suppose

> *Balls of different sizes and different weights are dropped from different heights from the Leaning Tower of Pisa in order to observe how the size, weight and height influence the time of descent.*

One may want to make sure that in this test:

> *For a fixed size, the weight of the object is independent of the height from which it is dropped.*

Ideally, the height and the weight would be made independent of each other, given the size. Then the test might indicate that the time of descent is, for fixed size, independent of the weight of the ball.

5. Richer independence

The speed of light is constant in vacuum but otherwise depends on the medium.

In a fixed medium, the speed of light is independent of the movement of the observer or the source.

5. Richer independence

We now give exact mathematical content to $\vec{y} \perp_{\vec{x}} \vec{z}$:
$\vec{y}$ is independent of $\vec{z}$, when $\vec{x}$ is kept fixed:

Definition

A team X satisfies the atomic formula $\vec{y} \perp_{\vec{x}} \vec{z}$ if for all $s, s' \in X$ such that $s(\vec{x}) = s'(\vec{x})$ there exists $s'' \in X$ such that $s''(\vec{x}) = s(\vec{x})$, $s''(\vec{y}) = s(\vec{y})$, and $s''(\vec{z}) = s'(\vec{z})$).

5. Richer independence

	x	y	z	u
s	5	6	10	
s'	5	13	18	

$$y \perp_x z$$

5. Richer independence

	x	y	z	u
s	5	6	10	
s'	5	13	18	
s''	5	6	18	
s'''	5	13	10	

$$y \perp_x z$$

5. Richer independence: Dependence can imply independence

Lemma

$=\!(\vec{x}, \vec{y})$ *logically implies* $\vec{y} \perp_{\vec{x}} \vec{z}$.

5. Richer independence: Only a constant can be independent of itself

Lemma

$\vec{y} \perp_{\vec{x}} \vec{z}$ *logically implies* $=\!(\vec{x}, \vec{y} \cap \vec{z})$.

5. Richer independence: Dependence is a special case of independence

Corollary

$=\!(\vec{x}, \vec{y}) \iff \vec{y} \perp_{\vec{x}} \vec{y}$

Lemma

1. $\vec{x} \perp_{\vec{x}} \vec{y}$ *(Reflexivity Rule)*
2. $\vec{z} \perp_{\vec{x}} \vec{y} \Rightarrow \vec{y} \perp_{\vec{x}} \vec{z}$ *(Symmetry Rule)*
3. $\vec{y} \perp_{\vec{x}} \vec{y} \Rightarrow \vec{y} \perp_{\vec{x}} \vec{z}$ *(Constancy Rule)*
4. $\vec{y}y' \perp_{\vec{x}} \vec{z}z' \Rightarrow \vec{y} \perp_{\vec{x}} \vec{z}$. *(Weakening Rule)*
5. *If* $\vec{z'}$ *is a permutation of* $\vec{z}$, $\vec{x'}$ *is a permutation of* $\vec{x}$, $\vec{y'}$ *is a permutation of* $\vec{y}$, *then* $\vec{y} \perp_{\vec{x}} \vec{z} \Rightarrow \vec{y'} \perp_{\vec{x'}} \vec{z'}$. *(Permutation Rule)*

Lemma

1. $\vec{z} \perp_{\vec{x}} \vec{y} \Rightarrow \vec{y}\vec{x} \perp_{\vec{x}} \vec{z}\vec{x}$ *(Fixed Parameter Rule)*
2. $\vec{x} \perp_{\vec{z}} \vec{y} \wedge \vec{u} \perp_{\vec{z}\vec{x}} \vec{y} \Rightarrow \vec{u} \perp_{\vec{z}} \vec{y}$. *(First Transitivity Rule)*
3. $\vec{y} \perp_{\vec{z}} \vec{y} \wedge \vec{x} \perp_{\vec{y}} \vec{u} \Rightarrow \vec{x} \perp_{\vec{z}} \vec{u}$ *(Second Transitivity Rule)*

Note that the Second Transitivity Rule gives by letting $\vec{u} = \vec{x}$:

$$\vec{y} \perp_{\vec{z}} \vec{y} \wedge \vec{x} \perp_{\vec{y}} \vec{x} \Rightarrow \vec{x} \perp_{\vec{z}} \vec{x},$$

which is the transitivity axiom of functional dependence. In fact Armstrong's Axioms are all derivable from the above rules. It remains open whether our rules permit a completeness theorem like Armstrong's Axioms do, and like we have for $x \perp y$.

1. $\vec{z} \perp_{\vec{x}} \vec{y} \Rightarrow \vec{y}\vec{x} \perp_{\vec{x}} \vec{z}\vec{x}$ (Fixed Parameter Rule) Since $\vec{x}$ is fixed, it does not generate new variety which would have to be taken care of.
2. $\vec{x} \perp_{\vec{z}} \vec{y} \wedge \vec{u} \perp_{\vec{z}\vec{x}} \vec{y} \Rightarrow \vec{u} \perp_{\vec{z}} \vec{y}$. (First Transitivity Rule) Suppose $s, s' \in X$ have the same z. There is $s'' \in X$ with the same z, but x from s and y from s'. So s, s'' agree about z and x. By the second assumption there is $s''' \in X$ which agrees with s, s'' on zx but picks u from s and y from s''. This is what we wanted.
3. $\vec{y} \perp_{\vec{z}} \vec{y} \wedge \vec{x} \perp_{\vec{y}} \vec{u} \Rightarrow \vec{x} \perp_{\vec{z}} \vec{u}$ (Second Transitivity Rule) By assumption, $\vec{z}$ determines $\vec{y}$. So if $\vec{x}$ and $\vec{u}$ are independent when $\vec{y}$ is kept fixed, the same holds if $\vec{z}$ is kept fixed.

6. Independence logic

First order logic:

- Atomic formulas $x = y$, $R(x_1, ..., x_n)$ with their negations.
- $\wedge, \vee, \exists, \forall$
- A formula $\phi(x_1, ..., x_n)$ tells about the individuals $x_1, ..., x_n$ in relation to each other and other unspecified individuals.
- Example: In this graph: x_1 and x_2 are not neighbors but some neighbor of x_1 is a neighbor of x_2.

6. Independence logic

Dependence Logic *Cambridge University Press 2007*

- Add new atomic formulas "x_n is completely determined by $x_1, ..., x_{n-1}$" to first order logic.
- $\wedge, \vee, \exists, \forall$
- A formula $\phi(x_1, ..., x_n)$ tells about mutual dependences of attributes $x_1, ..., x_n$ in a set of data.
- Example: In this data: x_3 is determined by x_1 except when $x_3 = x_2$.

6. Independence logic

Independence logic Grädel-V. 2010

- Add new atomic formulas "x_n is completely independent from $x_1, ..., x_{n-1}$" to dependence logic.
- $\wedge, \vee, \exists, \forall$
- A formula $\phi(x_1, ..., x_n)$ tells about mutual dependences and independences of attributes $x_1, ..., x_n$ in a set of data.
- Example: In this data: The time of descent is independent of the weight of the ball.

6. Independence logic: From equations to dependences

- Classical logic: equations $t \times t \times g = 2 \times h$
- Dependence logic: dependences $=\!(h, t)$
- Independence logic: independences $t \perp_h m$
- Each is a special case of the next.

6. Independence logic: Details

Definition

We define *independence logic* as the extension of first order logic by the new atomic formulas

$$\vec{y} \perp_{\vec{x}} \vec{z}$$

for all sequences $\vec{y}, \vec{x}, \vec{z}$ of variables. The negation sign $\neg$ is allowed in front of atomic formulas. The other logical operations are $\wedge, \vee, \exists$ and $\forall$. The semantics is defined for the new atomic formulas as above, and in other cases as for dependence logic.

Equivalent game-theoretic semantics: a winning strategy should allow mixing of plays in the same way as the above definition mixes assignments s and s' into a new one s''.

Cannot use signaling to go around demands of imperfect information.

Theorem

The expressive power of formulas $\phi(x_1,...,x_n)$ of dependence logic is exactly that of existential second order sentences with the predicate for the team negative. More exactly, let us fix a vocabulary L and an n-ary predicate symbol $S \notin L$. Then:

- *For every L-formula $\phi(x_1,...,x_n)$ of dependence logic there is an existential second order $L \cup \{S\}$-sentence $\Phi(S)$, with S negative only, such that for all L-structures M and all teams X:*

$$M \models_X \phi(x_1,...,x_n) \iff M \models \Phi(X). \tag{1}$$

- *For every existential second order $L \cup \{S\}$-sentence $\Phi(S)$, with S negative only, there exists an L-formula $\phi(x_1,...,x_n)$ of dependence logic such that (1) holds for all L-structures M and all teams $X \neq \emptyset$.*

Proposition

The expressive power of formulas $\phi(x_1, ..., x_n)$ of independence logic is contained in that of existential second order sentences with a predicate S for the team. More exactly, let us fix a vocabulary L and an n-ary predicate symbol $S \notin L$. Then for every L-formula $\phi(x_1, ..., x_n)$ of independence logic there is an existential second order $L \cup \{S\}$-sentence $\tau_\phi(S)$ such that for all L-structures M and all teams X: $M \models_X \phi(x_1, ..., x_n) \iff M \models \tau_\phi(X)$.

Corollary

For sentences independence logic and dependence logic are equivalent in expressive power.

Note that formulas of independence logic need not be closed downward, for example $x \perp y$ is not. This is a big difference to dependence logic. Still, the empty team satisfies every independence formula.

The sentence

$$\forall x \forall y \exists z (z \perp x \wedge z = y)$$

is valid in harmony with the intuition that the existential player should be able to make a decision to be independent of x when she chooses z whether she lets $z = y$ or not.

7. Expressive power

The sentence

$$\forall x \exists y \exists z (z \perp x \wedge z = x)$$

is not valid in harmony with the intuition that the existential player needs to follow what the universal player is doing with his x in order to be able to hit $z = x$. In independence friendly logic the sentence

$$\forall x \exists y \exists z / x (z = x),$$

is valid which is often found counter-intuitive. The trick (called "signaling") is that the existential player stores the value of x into y and then chooses z on the basis of y, apparently not needing to know what x is.

One may consider the entire independence friendly logic with the following interpretation:

$$[\exists x/\vec{y}\phi(x,\vec{y},\vec{z})]^* = \exists x(\vec{y} \perp_{\vec{z}} x \wedge [\phi(x,\vec{y},\vec{z})]^*)$$

As we have seen above this interpretation is not necessarily entirely faithful. However, the atom $\vec{y} \perp_{\vec{z}} x$ has one clearly distinguishable meaning of independence of $\vec{y}$ from x so it might be interesting to look at independence friendly logic with this interpretation.

7. Expressive power: Partially ordered quantifiers

Lemma

$$\begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} R(x,y,u,v) \iff \forall x \exists y \forall u \exists v (v \perp x \wedge R(x,y,u,v))$$

Independence logic is not so simple ...

Let κ be the smallest κ such that if ϕ is true in all models $< \kappa$, then ϕ is true in all models what so ever. $\kappa = \aleph_\kappa = \beth_\kappa$. If there are measurable cardinals, then κ is bigger than the first of them. It is always smaller than the first supercompact cardinal.

For first order logic $\kappa = \aleph_1$.

The main open question raised by the above discussion is the following, formulated for finite structures:

Open Problem: Characterize the NP properties of teams that correspond to formulas of independence logic.

Note that for dependence logic this is solved above: They are exactly those NP properties of teams that can be expressed in Σ^1_1 with a predicate that occurs only negatively.

8. Application: Social choice

- Individual voters are the variables.
- The values of these variables are the preference relations of the individuals.
- An assignment = a profile.
- The social choice function is just one variable.
- Arrowian axioms invoke dependence only, but the proof of Arrow's theorem depends on assumptions that invoke independence-type assumptions about the behaviour of the electorate.

9. Conclusion: Basic ideas

- Dependence on moves of the other player.
- Independence from the moves of the other player.
- What are the logical principles that these concepts follow (like identity follows identity axioms)?

9. Conclusion: Basic ideas

- We can add both dependence and independence of variables to first order logic.
- Mathematical—not syntactic—meaning.
- Dependence/independence: The essence of scientific discovery.
- Applies equally to the study of empirical data, where the laws are hidden.
- Generalizes game theoretic semantics.
- A perspective of the logic of interaction.

Thank you!