LINT workshop in Tampere Feb 2009: Observations on dependence logic

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Abstract

The basic concept of dependence logic [2] is

$$\mathfrak{M} \models_X \phi, \tag{1}$$

or just $X \models \phi$ if the underlying model $\mathfrak{M}$ is evident from the context. Intuitively, (1) means

> *Every $s \in X$ satisfies ϕ in the first order sense, and in addition, X manifests "dependence".*

The concept of dependence is fairly general, when one thinks of all the various uses of the word in ordinary life. The dependence concept embodied in dependence logic $\mathfrak{D}$ is of a particular kind. It starts from strong basic atomic dependences, which clearly manifest dependence, whatever one thinks of the concept. Then it builds more complex dependences by means of the logical operations $\neg, \vee, \wedge, \forall$ and $\exists$.

*Research partially supported by grant 40734 of the Academy of Finland.

The first type of atomic dependence is somewhat singular. A team X satisfies $t = t'$ for some terms t and t' iff every assignment s in X gives t and t' the same value (denoted $t\langle s\rangle$). So the dependence here is one of being identical. The second type of atomic dependence is: A team X satisfies $Rt_1...t_n$ iff the n-tuple $(t_1\langle s\rangle, ..., t_n\langle s\rangle)$ of values of the terms $t_1, ..., t_n$ under an assignment s is in the interpretation of the relation symbol R every s in X. The third type of atomic dependence declares dependence by total "obedience": A team X satisfies $=(t_1, ..., t_n)$ iff there is a function f such that the value of the term $t_n\langle s\rangle = f(t_1\langle s\rangle, ..., t_n\langle s\rangle)$ for all $s \in X$. This dependence is "dependence by slavery": the value of t_n depends on the values of $t_1, ..., t_n$ by the undoubtedly very strong way of being completely determined by them. Surprisingly, the singular form $=(t)$, i.e. the value of t is constant in X, is extremely useful.

$X \vDash \phi \wedge \psi$	iff	there are X_0 and X_1 such that $X_0 \vDash \psi$, $X_1 \vDash \psi$, and $X \subseteq X_0 \cap X_1$.
$X \vDash \phi \vee \psi$	iff	there are X_0 and X_1 such that $X_0 \vDash \psi$, $X_1 \vDash \psi$, and $X \subseteq X_0 \cup X_1$.
$X \vDash \forall x\phi$	iff	there is Y such that $Y \vDash \phi$ and for every $s \in X$ we have $s(a/x) \in Y$ for every $a \in M$.
$X \vDash \exists x\phi$	iff	there is Y such that $Y \vDash \phi$ and for every $s \in X$ we have $s(a/x) \in Y$ for some $a \in M$.

Equivalently, $X \models \phi \wedge \psi$ iff $X \models \phi$ and $X \models \psi$. The classical disjunction $X \models \phi \vee_{cl} \psi \iff X \models \phi$ or $X \models \psi$ can be defined from the above (in models with at least two elements):

$$\exists x \exists y (=(x) \wedge =(y) \wedge ((x = y \wedge \phi) \vee (\neg x = y \wedge \psi)).$$

There are two non-classical logical operations which have interesting applications in dependence logic ([1]):

$X \vDash \phi \supset \psi$	iff	for all $Y \subseteq X$: $Y \vDash \phi$ implies $Y \models \psi$.
$X \vDash \phi \multimap \psi$	iff	for all Y: $Y \vDash \phi$ implies $X \cup Y \models \psi$.

One of the interesting features of these connectives is the following pair of Galois connections:

$$\phi \wedge \psi \models \theta \quad \text{iff} \quad \phi \models \psi \supset \theta$$
$$\phi \vee \psi \models \theta \quad \text{iff} \quad \phi \models \psi \multimap \theta$$

We can think of dependence logic as a modal logic as follows: We think of the set of all teams (of a fixed structure) as a frame with the following accessibility relations:

$$\begin{array}{lll} R_\vee(X,Y,Z) & \text{iff} & X = Y \cup Z. \\ R_{\exists,x}(X,Y) & \text{iff} & \text{for every } s \in X \text{ we have} \\ & & s(a/x) \in Y \text{ for some } a \in M. \\ R_{\forall,x}(X,Y) & \text{iff} & \text{for every } s \in X \text{ we have} \\ & & s(a/x) \in Y \text{ for all } a \in M. \end{array}$$

Respectively, we have the following modal operations:

$$\begin{array}{lll} X \vDash \Diamond_\vee(\phi,\psi) & \text{iff} & \text{there are } X_0 \text{ and } X_1 \text{ such that} \\ & & X_0 \vDash \phi,\ X_1 \vDash \psi, \text{ and } R_\vee(X,X_0,X_1). \\ X \vDash \Diamond_{\forall,x}\phi & \text{iff} & \text{there is } Y \text{ such that } Y \vDash \phi \text{ and } R_{\forall,x}(X,Y) \\ X \vDash \Diamond_{\exists,x}\phi & \text{iff} & \text{there is } Y \text{ such that } Y \vDash \phi \text{ and } R_{\exists,x}(X,Y) \end{array}$$

The atomic propositions in such a modal logic would be propositional symbols corresponding to equations $t = t'$ and atomic relations $Pt_1...t_n$, plus special propositional symbols corresponding to the dependence atoms $=(t_1,...,t_n)$. For example, we have a propositional symbol $q_{x=y}$ for $x = y$ and a propositional symbol p_x for each $=(x)$. An example of such a modal sentence would be:

$$\Diamond_{\exists,x}\Diamond_{\exists,y}(p_x \wedge p_y \wedge \Diamond_\vee(q_{x=y} \wedge \phi, \neg q_{x=y} \wedge \psi)).$$

It would be interesting to exhibit relevant properties of the accessibility relations $R_\vee, R_{\exists,x}, R_{\forall,x}$ and study dependence logic as a modal logic in arbitrary Kripke structures with these relations. Can one prove decidability results, finite model properties, and develop correspondence theory?

Semantic values of formulas of dependence logic are sets $\{X : X \models \phi\}$ of teams and teams themselves are sets of assignments. Consider the following structure (S is an arbitrary set):

$$(\mathcal{P}(\mathcal{P}(S)), \otimes, \wedge), \tag{2}$$

where

$$\begin{array}{lcl} \mathcal{X} \otimes \mathcal{Y} & = & \{X \cup Y : X \in \mathcal{X}, Y \in \mathcal{Y}\}. \\ \mathcal{X} \wedge \mathcal{Y} & = & \mathcal{X} \cap \mathcal{Y}. \end{array}$$

What are the algebraic properties of such structures? Clearly, $\otimes$ and $\vee$ are commutative and associative, and $\wedge$ is idempotent ($\mathcal{X} = \mathcal{X} \wedge \mathcal{X}$), but in general

$$\begin{array}{lcl} \mathcal{X} \otimes \mathcal{X} & \neq & \mathcal{X}. \\ \mathcal{X} \wedge (\mathcal{Y} \otimes \mathcal{Z}) & \neq & (\mathcal{X} \wedge \mathcal{Y}) \otimes (\mathcal{X} \wedge \mathcal{Z}). \\ \mathcal{X} \otimes (\mathcal{Y} \wedge \mathcal{Z}) & \neq & (\mathcal{X} \otimes \mathcal{Y}) \wedge (\mathcal{X} \otimes \mathcal{Z}). \end{array}$$

If we take S in (2) to be the set of all assignments in a given structures, then $[\phi] = \{X : X \models \phi\}$ is an element of the algebra (2) and

$$\begin{array}{lcl} [\phi \vee \psi] & = & [\phi] \otimes [\psi]. \\ [\phi \wedge \psi] & = & [\phi] \wedge [\psi]. \end{array}$$

Thus algebraic properties of (2) immediately reflect logical properties of $\mathfrak{D}$. For sentences the set S contains only one element, namely the empty assignment $\emptyset$. In this case (2) consists of just four elements, namely $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$. Actually, the truth values $[\phi]$ are always non-empty (they contain the empty team as an element) downward closed sets, so we may replace (2) by

$$(M, \otimes, \wedge), \tag{3}$$

where

$$M = \{\mathcal{X} \in \mathcal{P}(\mathcal{P}(S)) : \emptyset \in \mathcal{X} \text{ and } \forall X \in \mathcal{X} \forall Y \subseteq X (Y \in \mathcal{X})\}.$$

In the case $S = \{\emptyset\}$ we get only two elements into M, namely $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$. In this sense $\mathfrak{D}$ is a two-valued logic. Let us denote $0 = \{\emptyset\}$ and $1 = \{\emptyset, \{\emptyset\}\}$. SO we have the algebra

$$(\{0, 1\}, \otimes, \wedge), \tag{4}$$

and this algebra is distributive. Moreover, $\otimes$ is idempotent. So in this special case we have a Boolean algebra, but as soon as we have free variables and the algebra gets bigger, we lose the Boolean structure.

Theorem 1 *For a variety of logics L^* the set $\mathfrak{D} \cap L^*$ is undecidable. (Joint work with V. Goranko)*

Theorem 2 *$\mathfrak{D}(\supset)$ and $\mathfrak{D}(\multimap)$ fail to have a variety of model-theoretic properties. (Joint work with S. Abramsky)*

References

[1] S. Abramsky and J. Väänänen. From IF to BI: A tale of dependence and separation. *Synthese*, to appear.

[2] Jouko Väänänen. *Dependence logic*, volume 70 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2007.