
A Modal Walk Through Space

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ABSTRACT. We investigate the major mathematical theories of space from a modal standpoint: topology, affine geometry, metric geometry, and vector algebra. This allows us to see new fine-structure in spatial patterns which suggests analogies across these mathematical theories in terms of modal, temporal, and conditional logics. Throughout the modal walk through space, expressive power is analyzed in terms of language design, bisimulations, and correspondence phenomena. The result is both unification across the areas visited, and the uncovering of interesting new questions.

RÉSUMÉ. HOW DO I REMOVE THE RESUME AND MOTCLES ???

KEYWORDS: spatial reasoning, modal logics, arrow logics, linear logic, conditional logic, topology, affine geometry, elementary geometry, mathematical morphology

MOTS-CLÉS : HOW DO I REMOVE THE RESUME AND MOTCLES ???

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1. Logics of Space

The study of Space has been a driving force in the foundations of mathematics, witness the history of formal proof, axiomatic theories, or the traditional geometric construction problems. In the development of logic, studies of space have been a much more marginal theme, with only scattered results. In particular, even though there is a well-established area of ‘temporal logic’, relevant to knowledge representation, process analysis, and other useful purposes, there is no similar ‘spatial logic’. Nevertheless, logical systems for spatial patterns are attracting growing attention these days—partly under the influence of computer science and AI, where spatial representation and image structure are now major themes. This paper is an attempt at putting this development in a broader perspective, showing in particular how modal structures arise across a variety of analysis of space. We do not offer new results, and we do not even pretend to give a complete survey.

Like Time, Space can be studied from many angles: mathematical, physical, linguistic, psychological. The literature has two broad approaches. One takes Space as something given, and studies its ontological structure: what are the primitive objects, and their relations? There are no unique answers here. Affine geometry, metric geometry, topology, linear algebra, each have their own take on spatial structures—and this proliferation continues today, e.g. in ‘mathematical morphology’ [MAT 67, SER 82]. Moreover, philosophers have added alternative theories to the mathematical mainstream, such as ‘mereology’—and they are still exploring spatial patterns that have not even been mathematized at all [CAS 99]. The role of ‘logic’ here is as in other branches of mathematics. One designs logical languages for describing constellations of objects and facts inside the relevant structures - say, a spatial pattern of regions—and one designs systems for valid reasoning about these—the appropriate ‘geometry’. This serves two purposes: (a) analysis of existing over-all theories in the area, and (b) design of natural ‘fragments’ striking a balance between expressive power and computational complexity. This fine-structure, in its modal guise, is the red thread in what follows.

The other approach does not start from some independently existing notion of Space, but rather from some existing human practice, e.g. a language with spatial expressions (say locative prepositions) or a diagrammatic way of visualizing things. One then determines minimal spatial structures that ‘fit’ in the sense of validating the practice. E.g., the repertoire of spatial expressions in natural languages can be seen as a record of how we normally position things in space—and a logical system can bring this out in some crisp manner. In particular, a logical semantics for such a system would provide ‘spatial patterns’, which may or may not be like the structures arising on the independent ontological study of space. This is not what we will do here, even though it is certainly a legitimate perspective, and one which is close to important current studies of spatial expressions, maps, and diagrams [HAM 95, KER 01].

Using a logic to study space buys into a typical *modus operandi*. This involves language design and semantic expressiveness, structural simulations between models

that fit the language, general logical validities for the chosen type of structure, special purpose axioms corresponding to special structures, complexity of various logical tasks (model checking, similarity testing, satisfiability) and more methodologically, translations between different systems addressing the same structures. In addition to these general concerns, the field of Temporal Logic [BEN 83b, GAB 94] suggests further themes for a study of Space, from the longer experience in studying Time. These logical issues are not standard features in the usual mathematical theories of space, and their added value must show in practice. Conversely, in doing so, one must see what spatial content can be attached to abstract logical calculations. Even without exploring all these features in exhaustive detail, we do think the following survey shows that all these promises can be made good in a modal perspective.

2. Topology

The coarsest mathematical view of Space is only concerned with insides, exteriors, and boundaries of regions. Topology was developed in the early 20th century, as an elegant generalization of recurrent features of those spatial structures that survive elastic deformations. A topological space is any pair (X, O) , where O is a family of subsets of X containing the empty set and X itself, which is closed under finite intersections and arbitrary unions. The structure-preserving notion of similarity between topological spaces are continuous maps, known from Analysis in their $\epsilon - \delta$ formulation, whose definition needs nothing but open sets. The resulting theory is very general: an apparent weakness which, as so often, is really a strength. Topological structures arise in a much broader arena than space, including patterns of computation and information. It was observed in the 1930s that the modal logic **S4** describes a small part of all this, viz. the basic algebra of interior and closure. This is the ‘topological semantics’ for modal logic, a phrase that makes the language primary, and the spatial interpretation secondary. Here, our interest is the other way around. In this first section, we review some relevant results, showing how modal logics can serve as weaker or stronger topological theories of space.

2.1. The basic modal language of topology

Let us begin with the basic definitions of the topological interpretation of modal logics on the language **S4**, [TAR 38, MCK 44]. The language is composed, as usual, of a countable set of proposition letters, boolean connectives $\neg, \vee, \wedge, \rightarrow$, and modal operators \Box, \Diamond . A model is a topological space equipped with a valuation function $\nu : P \rightarrow \mathcal{P}(X)$, where P is the set of proposition letters. A structure $M = \langle (X, O), \nu \rangle$, where (X, O) is a topological space, is called a *topo-model*. Here is the precise semantic definition (in modern modal logic terms).

Definition 1 (basic topological semantics) Truth of modal formulas is defined inductively at points x in a topological model M :

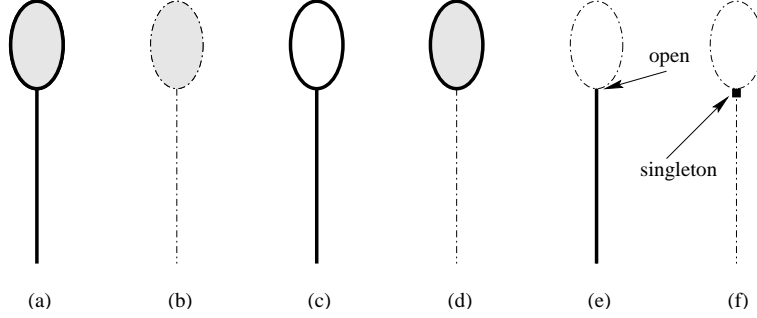


Figure 1. A formula of **S4** identifies a region in a topological space. (a) a spoon, p . (b) the container part of the spoon, $\Box p$. (c) the boundary of the spoon, $\Diamond p \wedge \Diamond \neg p$. (d) the container part of the spoon with its boundary, $\Diamond \Box p$. (e) the handle of the spoon, $p \wedge \neg \Diamond \Box p$. In this case the handle does not contain the junction point handle-container. (f) the joint point handle-container of the spoon, $\Diamond \Box p \wedge \Diamond (p \wedge \neg \Diamond \Box p)$: a singleton in the topological space.

$M, x \models p$	iff	$x \in \nu(p)$ (with $p \in P$)
$M, x \models \neg \varphi$	iff	not $M, x \models \varphi$
$M, x \models \varphi \wedge \psi$	iff	$M, x \models \varphi$ and $M, x \models \psi$
$M, x \models \Box \varphi$	iff	$\exists o \in O : x \in o \wedge \forall y \in o : M, y \models \varphi$
$M, x \models \Diamond \varphi$	iff	$\forall o \in O : \text{if } x \in o, \text{ then } \exists y \in o : M, y \models \varphi$

As usual we can economize by defining, e.g., $\varphi \vee \psi$ as $\neg(\neg\varphi \wedge \neg\psi)$, and $\Diamond \varphi$ as $\neg \Box \neg \varphi$. We will do this whenever convenient.

Each formula of the language denotes a region of the topological space being modeled. For instance, take the real plane \mathbb{R}^2 with the standard topology. Consider a valuation function sending a spoon shape region (and only that region) to the propositional letter p , Figure 1.a. Then, the formula $\neg p$ denotes the region not occupied by the spoon, i.e., the background. The formula $\Box p$ denotes the interior of the spoon region p and so on, as in Figure 1.

The topological interpretation brings a noticeable shift in perspective from the usual possible worlds semantics. E.g., *locality* now means that a formula is true at M, x iff it is true at any open neighborhood of x in M (viewed as a sub-model). Thus, *regions* are essential, and more generally, a modal approach provides a calculus of regions de-emphasizing constellations of points. As such, it is close to ‘region versus points’ movements in theories of time and space, [ALL 83, BEN 83b, ALL 85, RAN 92].

To understand the expressive power of a modal language a suitable notion of bisimulation is needed. The following definition reflects the semantic definition of

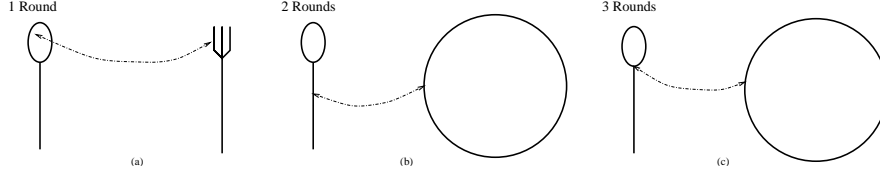


Figure 2. Game rounds needed for distinguishing shapes.

the modal operators and can be seen as composed of two sub-moves: one in which points are related and one in which containing opens are matched.

Definition 2 (topological bisimulation) A *topological bisimulation* between two topological models $\langle X, O, \nu \rangle$, $\langle X', O', \nu' \rangle$ is a non-empty relation $\rightleftharpoons \subseteq X \times X'$ such that, if $x \rightleftharpoons x'$, then:

- (i) $x \in \nu(p) \Leftrightarrow x' \in \nu'(p)$ (for any proposition letter p)
- (ii) (*forth*): $x \in o \in O \Rightarrow \exists o' \in O' : x' \in o' \text{ and } \forall y' \in o' : \exists y \in o : y \rightleftharpoons y'$
- (iii) (*back*): $x' \in o' \in O' \Rightarrow \exists o \in O : x \in o \text{ and } \forall y \in o : \exists y' \in o' : y \rightleftharpoons y'$

A bisimulation is *total* if its domain is X and its range X' . If only the atomic clause (i) and the forth condition (ii) hold, we say that the second model *simulates* the first.

Topo-bisimulation indeed captures the adequate notion of ‘model equivalence’ for **S4** topologically interpreted. Evidence for this comes from two results like the following, cf. [AIE 02c].

Theorem 1 Let $M = \langle X, O, \nu \rangle$, $M' = \langle X', O', \nu' \rangle$ be two models, $x \in X$, and $x' \in X'$ two bisimilar points. Then, for any modal formula φ , $M, x \models \varphi$ iff $M', x' \models \varphi$. In words, modal formulas are invariant under bisimulations.

Theorem 2 Let $M = \langle X, O, \nu \rangle$, $M' = \langle X', O', \nu' \rangle$ be two finite models, $x \in X$, and $x' \in X'$ such that for every φ , $M, x \models \varphi$ iff $M', x' \models \varphi$. Then there exists a bisimulation between M and M' connecting x and x' . In words, finite modally equivalent models are bisimilar.

Topo-bisimulations are model theoretic tools for assessing expressivity of our language with respect to spatial patterns. Nevertheless, when comparing e.g. two image representations, it may still be too coarse. To refine the similarity matching, one can define a topological model comparison game $TG(X, X', n)$ between two models X, X' in the Ehrenfeucht-Fraïssé spirit. The idea of the game is that two players challenge each other picking elements from the two models to compare. One player wins if he can show the models to be different, the other wins if he can show the models to be ‘similar’. Winning strategies for the similarity player ‘Duplicator’ in

infinite games yield topo-bisimulations. Furthermore, for finite-length games, games and modal formulas are connected by the Adequacy Theorem:

Theorem 3 *Duplicator has a winning strategy in $TG(X, X', n, x, x')$ iff x and x' satisfy the same formulas of modal operator depth up to n in their respective models X, X' .*

The formal definition of a game, discussion of plays and strategies (cf. Figure 2) are in [AIE 02c], while the use of topo-games to compare models deriving from image descriptions is illustrated in [AIE 02b]. The first paper also points out that topo-bisimulations are coarse versions of more traditional topological mappings.

Theorem 4 *Let $\langle X, O, \nu \rangle$ and $\langle X', O', \nu \rangle$ be two topological models. If $\langle X, O \rangle$ and $\langle X', O' \rangle$ are homeomorphic, then there exist a non-trivial topo-bisimulation between the topo-models.*

A trivial bisimulation, i.e., one in which all points of a model are related to all points of the other model, always exists. By non-trivial bisimulation in the previous theorem, we mean one in which the points are related following the homeomorphic mapping. See [AIE 02c] for a proof, as well as connections with other topological notions of similarity, such as homotopy.

Now consider logical validity, and hence the general calculus for spatial reasoning in this language. The logic **S4** is defined by the KT4 axioms and the rules of Modus Ponens and Necessitation. In the topological setting, the key principles are as follows, with an informal explanation added:

- | | |
|---|---|
| $\Box \top$ | (N) the whole space X is open |
| $(\Box \varphi \wedge \Box \psi) \leftrightarrow \Box(\varphi \wedge \psi)$ | (R) open sets are closed under finite intersections |
| $\Box \varphi \rightarrow \Box \Box \varphi$ | (4) idempotence of the interior operator |
| $\Box \varphi \rightarrow \varphi$ | (T) the interior of any set is contained in the set |

Using this set of axioms, the rules of inference are modus ponens and monotonicity of the modal box. The universally valid formulas topologically interpreted are precisely the theorems of **S4**. But [MCK 44] proved a much more striking result.

Theorem 5 ***S4** is complete for any metric space without isolated points.*

Thus, **S4** is also the complete logic of any space \mathbb{R}^n with the standard topology. [MIN 98] proved completeness for the Cantor set in a particularly elegant manner.

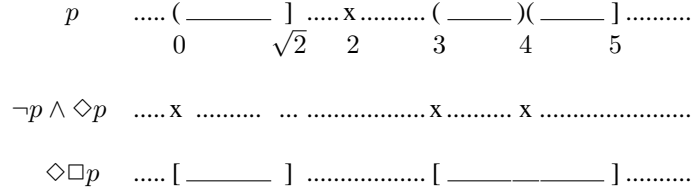


Figure 3. A serial set of \mathbb{R} and the defined sub-formulas by the axiom \mathbf{BD}_2 .

By adding extra constraints it is possible to identify specific spatial structures of interest. Take, for instance, the real line \mathbb{R} with the standard topology and consider the additional axioms:

$$\begin{aligned} \square(\square(p \rightarrow \square p) \rightarrow p) &\rightarrow p && \mathbf{(Grz)} \\ (\neg p \wedge \diamond p) &\rightarrow \diamond \square p && \mathbf{(BD}_2) \\ \neg(p \wedge q \wedge \diamond(p \wedge \neg q) \wedge \diamond(\neg p \wedge q) \wedge \diamond(\neg p \wedge \neg q)) &&& \mathbf{(BW}_2) \end{aligned}$$

These are complete with respect to the *serial* sets of the real line, being the finite unions of convex intervals [AIE 01]. To give an impression of what is going on, referring to Figure 3, consider the serial set denoted by p and the axiom \mathbf{BD}_2 . The axiom, which in Kripke semantics bounds the depth of the model to 2, in topological semantic states that the points that are both in the complement $\neg p$ of a region and in its closure $\diamond p$, must be in the regular closed portion $\diamond \square p$ of the region itself.

Similarly, one can look at interesting bi-dimensional topological spaces. Here is an axiom valid for \mathbb{R}^2 :

$$\diamond(\square p_3 \wedge \diamond(\square p_2 \wedge \diamond \square p_1 \wedge \neg p_1) \wedge \neg p_2) \rightarrow p_3 \quad \mathbf{(BD}_3)$$

The axiom, which in Kripke semantics bounds the depth of the model to 3, expresses a property of the ‘rectangular serial’ sets of the plane (again involving the boundary points of sets, as in the mono-dimensional case of \mathbf{BD}_2). These special structures are investigated in [AIE 01] and [BEN 02]. The latter provides axiomatizations for logics of this sort for real spaces of any dimension.

2.2. Extended modal languages

An extremely useful technique in modal logics gains expressive power without losing decidability, by adding a modal operator. For instance, if one needs to express notions connected to equality of states in Kripke semantics, one may add a difference operator $D\varphi$ which reads “there is a state different from the current one that satisfies φ .” The same move makes sense for space. Topological relations not captured by the basic modal language can be safely expressed by adding appropriate new modal operators. We have entered the realm of extended modal languages, see [RIJ 93, BEN 91b].

2.2.1. Universal reference and global properties

The basic language **S4** interpreted on topological spaces has a ‘local’ view of the world. A global perspective comes from the addition of a universal modality that expresses accessibility to any point [GOR 92]. Universal modalities were brought to the spatial reasoning community in [BEN 95]. For this purpose, one adds:

$$\begin{aligned} M, x \models E\varphi & \text{ iff } \exists y \in X : M, y \models \varphi \\ M, x \models U\varphi & \text{ iff } \forall y \in X : M, y \models \varphi \end{aligned}$$

More systematically the relevant new valid principles are those of **S5**:

$$\begin{aligned} E\varphi & \leftrightarrow \neg U\neg\varphi & \text{(Dual)} \\ U(\varphi \rightarrow \psi) & \rightarrow (U\varphi \rightarrow U\psi) & \text{(K)} \\ U\varphi & \rightarrow \varphi & \text{(T)} \\ U\varphi & \rightarrow UU\varphi & \text{(4)} \\ \varphi & \rightarrow UE\varphi & \text{(B)} \end{aligned}$$

In addition, the following ‘connecting’ principle is part of the axioms:

$$\diamond\varphi \rightarrow E\varphi$$

Using these principles, we notice that **S4_u** allows a normal form.

Proposition 6 *Every formula of **S4_u** is equivalent to one without nested occurrences of E, U .*

The definition of topo-bisimulation extends straightforwardly. It merely demands that bisimulations be *total* relations.

Theorem 7

- Extended modal formulas in **S4_u** are invariant under total bisimulations.
- Finite **S4_u**-modally equivalent models are totally bisimilar.

In a topological setting, fragments of this language can also be relevant. E.g., a continuous map has only one of the zig-zag clauses of topo-bisimulations. Now, consider ‘existential’ modal formulas constructed using only atomic formulas and their negations, \wedge, \vee, \square, E and U .

Corollary 8 *Let the simulation \rightarrow run from M to M' , with $x \rightarrow x'$. Then, for any existential modal formula φ , $M, x \models \varphi$ only if $M', x' \models \varphi$. In words, existential modal formulas are preserved under simulations.*

Here is an example, cf. [SHE 99, AIE 02c]. A topological space is *connected* if the only two sets that are both open and closed are the empty set and whole space itself. This property is expressible in **S4_u** in the following way:

$$U(\diamond p \rightarrow \square p) \rightarrow Up \vee U\neg p \tag{1}$$

RCC	$\mathbf{S4}_u$	Description
$\text{DC}(A, B)$	$\neg E(A \wedge B)$	A is disconnected from B
$\text{EC}(A, B)$	$E(\diamond A \wedge \diamond B) \wedge \neg E(\Box A \wedge \Box B)$	A and B are externally connected
$\text{P}(A, B)$	$U(A \rightarrow B)$	A is part of B
$\text{EQ}(A, B)$	$U(A \leftrightarrow B)$	A and B are equal

Figure 4. Expressing RCC relations via $\mathbf{S4}_u$.

Alternatively, there do not exist two disjoint open sets whose union covers the whole space. Again we can express this in $\mathbf{S4}_u$:

$$U(\Box p \vee \Box q) \wedge Ep \wedge Eq \rightarrow E(p \wedge q) \quad (2)$$

Now, here is a typical mathematical fact which gets a logical twist: connectedness of topological spaces is preserved under continuous surjections. Why? Well, consider a non connected space X' such that $\varphi = U(\Box \neg p \vee \Box p) \wedge Ep \wedge E\neg p$ is true at some point x . Let f be any continuous map from X to X' . Copying p to X with f yields a total simulation from X' to X , which preserves the existential modal formula φ . Then X is not connected either. Eventually, this observation can be used assess the preservation behavior of continuous maps much more generally.

By encoding a fragment of the Region Connection Calculus (RCC)[RAN 92] in the language $\mathbf{S4}_u$, Bennett showed the power of the language in expressing spatial arrangement of regions. The relevant elementary relations between regions that one can express are those of parthood and connectedness. The encoding is reported in Figure 4, which is the basis for the appropriate calculus in computer science and AI.

2.2.2. UNTIL a boundary

Another source of inspiration for extension of the expressive power of the basic language of topology comes from temporal formalisms. Consider the Since and Until logic of [KAM 68]. If one abstracts from the temporal behavior and interprets the modality in spaces with dimensionality greater than one, one gets an operator expressing something to be valid *up to a certain boundary* region, a sort of fence surrounding the current region. Here is a natural notion of spatial ‘Until’ in topological models:

$$M, x \models \varphi \mathcal{U} \psi \quad \text{iff} \quad \exists A : O(A) \wedge x \in A \wedge \forall y \in A. \varphi(y) \wedge \\ \forall z (z \text{ is on the boundary of } A \rightarrow \psi(z))$$

Defining the dual modality $\varphi\mathcal{U}^D\psi$ as usual is $\neg(\neg\varphi\mathcal{U}\neg\psi)$ we get:

$$M, x \models \varphi\mathcal{U}^D\psi \quad \text{iff} \quad \forall A : O(A) \wedge x \in A \rightarrow (\exists y \in A. \varphi(y) \vee \exists z(z \text{ is on the boundary of } A \wedge \psi(z)))$$

Using the notation of the basic modal language, we recall the topological definition of boundary of a set A :

$$\text{boundary}(A) = \diamond A \wedge \diamond \neg A$$

A graphical representation of the Until operator is presented in Figure 5. Its expressivity is richer than that of the basic modal language of space. E.g., one can express global properties inside connected components:

$$\mathcal{U}\varphi \perp \text{ iff some open component around the current point is all } \varphi$$

In connected spaces, this is equivalent to the universal modality U .

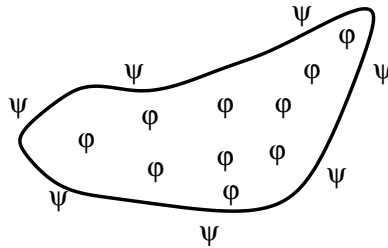


Figure 5. The region involved in $\varphi\mathcal{U}\psi$.

Which temporal principles valid in \mathbb{R} survive the move to more than one dimension? We do not provide a full axiomatization, but rather look at how temporal axioms

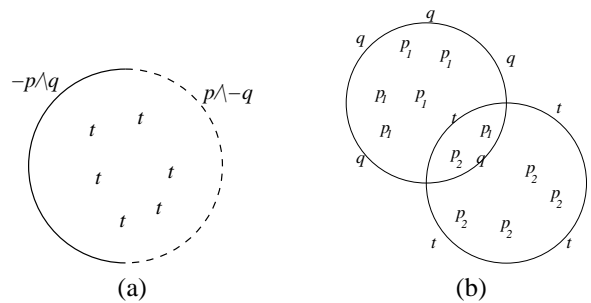


Figure 6. Examples of Until models.

behave in space and which new ones may arise. Two useful equivalences for obtaining normal forms in the one dimensional case are

$$\begin{aligned} t\mathcal{U}(p \vee q) &\leftrightarrow (t\mathcal{U}p) \vee (t\mathcal{U}q) \\ (p \wedge q)\mathcal{U}t &\leftrightarrow (p\mathcal{U}t) \wedge (q\mathcal{U}t) \end{aligned}$$

In our spatial setting, the first equivalence fails: Figure 6.a refutes the implication \rightarrow . But the other direction remains a valid principle of monotonicity. As for the second equivalence, its direction \rightarrow is a general monotonicity principle again. Conversely, we even have a stronger valid law:

$$p_1\mathcal{U}q \wedge p_2\mathcal{U}t \rightarrow (p_1 \wedge p_2)\mathcal{U}(q \vee t)$$

In [AIE 02a] there is a more sustained analysis of the spatial content of the \mathbb{R} complete Until logic of [BUR 84], with an auxiliary operator G :

$$Gp \leftrightarrow p\mathcal{U}\perp$$

Here is the set of axioms found in [BUR 84]:

$$G(p \rightarrow q) \rightarrow ((r\mathcal{U}p) \rightarrow (r\mathcal{U}q)) \wedge ((p\mathcal{U}r) \rightarrow (q\mathcal{U}r)) \quad (3)$$

$$p \wedge (r\mathcal{U}q) \rightarrow (r\mathcal{U}(q \wedge (r\mathcal{S}p))) \quad (4)$$

$$(q\mathcal{U}p) \leftrightarrow ((q \wedge (q\mathcal{U}p))\mathcal{U}p) \leftrightarrow q\mathcal{U}(q \wedge (q\mathcal{U}p)) \quad (5)$$

$$((q\mathcal{U}p) \wedge \neg(r\mathcal{U}p)) \rightarrow q\mathcal{U}(p \wedge \neg r) \quad (6)$$

$$((q\mathcal{U}p) \wedge (s\mathcal{U}r)) \rightarrow (((q \wedge s)\mathcal{U}(p \wedge r)) \vee ((q \wedge s)\mathcal{U}(p \wedge s)) \vee ((q \wedge s)\mathcal{U}(q \wedge r))) \quad (7)$$

Determining which Until laws survive from the real line to other spaces \mathbb{R}^n is a good illustration of the general issue of ‘transfer’ of temporal logic principles to spatial settings. Finally, as for topo-bisimulations for this richer language, we would need an extension of the proposals in [KUR 97] for dealing with the $\exists\forall$ -complexity of the truth condition for the spatial Until.

2.3. Standard logical analysis

The modal hierarchy of topological languages has a common root. All operators given have truth conditions in a second-order language quantifying over both points and sets of points. E.g., $\Box p$ says that $\exists A : O(A) \wedge x \in A \wedge \forall y : y \in A \rightarrow P(y)$. This language has the following vocabulary:

$\forall x$	quantification over points
$\forall A$	quantification over sets of points
$x = y$	identity
$x \in A$	membership of points in sets
$O(A)$	predicate of openness of sets

All fundamental topological notions are definable in this formalism. Here are two relevant observations.

Fact 1 Formulas of the second-order language without free predicate variables are preserved under topological homeomorphisms.

The proof is a simple induction.

Fact 2 All topological separation axioms T_i (with $0 \leq i \leq 4$) are expressible in the second-order language.

For example, one can express the T_2 axiom (defining the Hausdorff spaces) in the following way:

$$\forall x, y : (x \neq y \rightarrow \exists A, B : O(A) \wedge O(B) \wedge \neg \exists z (z \in A \wedge z \in B) \wedge x \in A \wedge y \in B)$$

Of course, this strong language has various much more tractable fragments, and the goal in ‘modal topology’ is finding these. But the second-orderness in this analysis maybe somewhat spurious. One can see this by a little ‘deconstruction’. The interior modality $\Box\varphi$ mixes elements of different sorts. $\Box\varphi$ is true in a point x if there exists an open set containing the point x itself and such that all points of the set satisfy φ . An alternative take would separate points and open sets into two separate modal quantifiers. The resulting modal logic was studied in [DAB 96] and in [GEO 93]. The main motivation for this was mainly an analysis of knowledge, but the authors mention it as a potential tool for visual reasoning. In this system, $\Box\varphi$ is defined as $\langle s \rangle [p]\varphi$, which is $M, x, o \models \exists o' \subseteq o \in O : x \in o' \wedge \forall y \in o' : M, y, o \models \varphi$.

Disregarding these epistemic concerns, one can take a plain two-sorted approach, just as in a general interpretation for the above second-order language [BEN 83c].

$$M, x \models \langle S \rangle \varphi \text{ iff } \exists A : x \in A \wedge M, A \models \varphi$$

$$M, A \models \langle p \rangle \varphi \text{ iff } \exists x : x \in A \wedge M, x \models \varphi$$

Here, models have two sorts of objects, namely *points* and *sets*, where the latter need not exhaust the whole power set of the former. This allows for ordinary bisimulations, suitably modified to link just the same sorts of objects across models. Still more closely to the basic modal language, one can also think of the second sort as *open sets* where the original topological $\Box\varphi$ becomes $\langle A \rangle [p]\varphi$, with A just ranging over open sets.

Under this decomposition, the base logic of topological space is no longer **S4**. E.g., reflexivity $\Box\varphi \rightarrow \varphi$ becomes

$$\langle A \rangle [p]\varphi \rightarrow \varphi$$

which expresses the fact that the accessibility relation between points and sets are inverses. Likewise transitivity becomes

$$\langle A \rangle [p]\varphi \rightarrow \langle A \rangle [p]\langle A \rangle [p]\varphi,$$

which follows from

$$[p]\varphi \rightarrow [p]\langle A \rangle [p]\varphi$$

which is simply a minimally valid consequence of conversion ($\psi \rightarrow [p]\langle A \rangle \psi$). The minimal logic of points and arbitrary sets related by membership is just the basic tense logic for a relation and its converse. Further properties of topologies, such as intersection closure and opens, become additional axioms on the propositional constant of defining open sets. Their most perspicuous formulation requires an extended modal language. For further details, we refer to [AIE 02a].

Either way, whether second-order or two-sorted first-order, there is a landscape of possible modal languages for topological patterns whose nature is by no means understood. For instance, one would like to understand what are natural well-chosen languages for simulations, and also, what are the *complexity jumps* between languages and their logics in this spectrum.

2.4. Related literature

Two lines of research intersect in this section on modal logics of topology. One is purely mathematical and logical, the other one more philosophical, with a later artificial intelligence twist. The former originated with the work of Tarski, later together with McKinsey, on semantics of the modal logic **S4** [TAR 38, MCK 44]. This was the first completeness proof for this logic, predating possible worlds semantics. But this spatial interpretation became a side-track, except for [RAS 63].

The other line of research has even older origins. Philosophers, such as Whitehead, were interested in formalizing fundamental notions like parthood, [WHI 29, LES 83]. This was the seed for what would later be called *mereotopology*, the theory of parthood and connectedness for regions, where mathematical foundations were developed, e.g., by [CLA 81] or by [ASH 95]. Mereotopology also captured the attention of the artificial intelligence community. [RAN 92] developed an influential system of regions based on connection and parthood: the *region connection calculus* (RCC). The decidable encoding for RCC in **S4_u** found in [BEN 95], was the crucial point where topological modal logic and the mereotopological road to spatial representation and reasoning really crossed.

3. Affine Geometry

Extending the expressive power of a modal logic of space may go beyond mere logical power. One can also enrich *geometrical power* by endowing spaces with more structure. A first elementary example is the property of a point being in the convex closure of a set. That is, there exists a segment containing the point, whose end-points are in the set. The notion of convexity is very important in many fields related to space (e.g., computational geometry [PRE 85]), but also in abstract cognitive settings

(e.g., conceptual spaces [GÄR 00]). Capturing convexity modally involves a standard similarity type, that of frames of points with a ternary relation of betweenness:

$$M, x \models C\varphi \text{ iff } \exists y, z : M, y \models \varphi \wedge M, z \models \varphi \wedge x \text{ lies in between } y \text{ and } z \quad (8)$$

This definition is slightly different from the usual notion of convex closure. It is a *one-step convexity* operator whose countable iteration yields the standard convex closure. The difference between the two definitions is visible in Figure 15. On the left are three points denoting a region. The standard convex closure operator gives the full triangle depicted on the right. The one-step convexity, on the other hand, gives the frame of the triangle and only when applied twice yields the full triangle. Another illustration of is presented in Figure 7. One-step convexity exhibits a modal pattern for an existential binary modality:

$$\exists yz : \beta(yxz) \wedge \varphi(y) \wedge \varphi(z)$$

where $\beta(yxz)$ is a ternary relation meaning that the point x is between the points y and z , Figure 15. From now on, we shall use the term convexity operator to refer to the one-step convexity operator defined in (8).

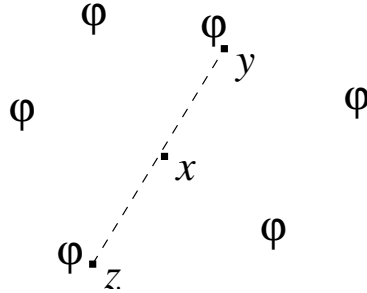


Figure 7. The point x is in the one-step convex closure φ .

3.1. Basic Geometry

Geometrical modal logic starts from standard bits of mathematics, viz. affine geometry, [BLU 61]. For later reference, here are the affine base axioms in a language with two sorts for points and lines, and an incidence relation as presented by Goldblatt [GOL 87]:

- A1** Any two distinct points lie on exactly one line.
- A2** There exist at least three non-collinear points.
- A3** Given a point a and a line L , there is exactly one line M that passes through a and is parallel to L .

There are also some properties that further classify affine planes. In particular, an affine plane is *Pappian* if every pair of its lines has the Pappus property:

A pair L, M of lines in an affine plane has the Pappus property if whenever a, b, c is a triple of points on L , and a', b', c' is a triple on M such that ab' is parallel to $a'b$ and ac' is parallel to $a'c$, then $b'c$ is parallel to bc' .

Affine spaces have a strong modal flavor, as shown by [BAL 97, BAL 98, VEN 99, STE 00], where two roads are taken. One merges points and lines into one sort of pairs $\langle \text{point, line} \rangle$ equipped with two incidence relations. The other has two sorts for points and lines, and a matching sorted modal operator.

But there are more expressive classical approaches to affine structure. In [TAR 59], Tarski gave a full first-order axiomatization of elementary geometry in terms of a ternary betweenness predicate $\beta(xyz)$, interpreted as Y lies between x and z , and quaternary equidistance $\delta(xyzu)$, interpreted as x is as distant from y as z is from u . We display it here as an ‘upper limit’, using Tarski’s original names for the axioms:

- A1** $\forall xy(\beta(xyx) \rightarrow (x = y))$, identity axiom for betweenness.
- A2** $\forall xyzu(\beta((xyu) \wedge \beta(yzu)) \rightarrow \beta(xyz))$, transitivity axiom for betweenness,
- A3** $\forall xyzu(\beta(xyz) \wedge \beta(xyu) \wedge (x \neq y) \rightarrow \beta(xzu) \vee \beta(xuz))$ connectivity axiom for betweenness,
- A4** $\forall xy(\delta(xyyx))$, reflexivity axiom for equidistance,
- A5** $\forall xyz(\delta(xyzz) \rightarrow (x = y))$, identity axiom for equidistance,
- A6** $\forall xyzuvw(\delta(xyzu) \wedge \delta(xyvw) \rightarrow \delta(zuvw))$, transitivity axiom for equidistance,
- A7** $\forall txyzu\exists v(\beta(xtu) \wedge \beta(yuz) \rightarrow \beta(xvy) \wedge \beta(ztv))$, Pasch’s axiom,
- A8** $\forall txyzu\exists vw(\beta(xut) \wedge \beta(yuz) \wedge (x \neq u) \rightarrow \beta(xzv) \wedge \beta(xyw) \wedge \beta(vtw))$, Euclid’s axiom,
- A9** $\forall xx'yy'zz'u'u'(\delta(xyx'y') \wedge \delta(yzy'z') \wedge \delta(xux'u') \wedge \delta(yuy'u') \wedge \beta(xyz) \wedge \beta(x'y'z') \wedge (x \neq y) \rightarrow \delta(zuz'u'))$, five-segment axiom,
- A10** $\forall xyuv\exists z(\beta(xyz) \wedge \delta(yzuv))$, axiom of segment construction,
- A11** $\exists xyz(\neg\beta(xyz) \wedge \neg\beta(yzx) \wedge \neg\beta(zxy))$, lower dimension axiom,
- A12** $\forall xyzuv(\delta(xuxv) \wedge \delta(yuyv) \wedge \delta(zuzv) \wedge (u \neq v) \rightarrow \beta(xyz) \vee \beta(yzx) \vee \beta(zxy))$, upper dimension axiom,
- A13** *All sentences of the form $\forall vw \dots (\exists z \forall xy(\varphi \wedge \psi \rightarrow \beta(zxy)) \rightarrow \exists u \forall xy(\varphi \wedge \psi \rightarrow \beta(xuy)))$, where φ stands for any formula in which the variables x, v, w, \dots , but neither y nor z nor u , occur free, and similarly for ψ , with x and y interchanged, elementary continuity axioms.*

Why is this beautiful complete and decidable axiomatization not all one wants to know? From a modal standpoint, there are two infelicities in this system. The axioms are too powerful, and one wants to look at more tractable fragments. First, the axiom system has high complexity. But also, the axioms mix betweenness and equidistance—whereas one first wants to understand affine and metric structure separately.

3.2. The general logic of betweenness

Our choice of primitives for affine space is again betweenness, where $\beta(xyz)$ means that point y lies in between x and z , allowing y to be one of these endpoints. Line structure is available by defining *collinearity* in terms of betweenness:

$$xyz \text{ are collinear iff } \beta(xyz) \vee \beta(yzx) \vee \beta(zxy)$$

‘Geometrical extensions’ of this sort can even define ‘extended modalities’, i.e., ‘logical extensions’ in our earlier terminology. Define first a binary betweenness modality $\langle B \rangle$:

$$M, x \models \langle B \rangle(\varphi, \psi) \quad \text{iff} \quad \exists y, z : \beta(yxz) \wedge M, y \models \varphi \wedge M, z \models \psi$$

Then, one can define the *existential* modality “at some point:”

$$E\varphi \text{ iff } \langle B \rangle(\varphi, \top) \tag{9}$$

This will work provided we require betweenness to satisfy:

$$\forall x \forall y \beta(xxy).$$

Without this, the defined modality will just range over the connected component of the current point of evaluation.

Natural specific structures on which to interpret our modal language include the \mathbb{R}^n for any n . But affine spaces really form a much more general class of structures. What are natural general frame conditions constraining these? As one does for temporal logics, the *universal first-order theory* of ordinary real space suggests good candidates. Consider just the betweenness part of Tarski’s elementary geometry. Axioms **A1-A3** for identity, transitivity, and linearity are all plausible as general affine properties. They are not sufficient, though, as one also wants some obvious variants of transitivity and linearity with points in other positions stated explicitly. A complete first-order axiomatization of pure affine elementary geometry was given in [SZC 65]. With Tarski, the latter are theorems, but their proofs go through other axioms involving equidistance. Further universal first-order assertions that hold in real space would express *dimensionality* of the space, which does not seem a plausible constraint in general.

At the next level of syntactic complexity, one then finds existential axioms and universal-existential ones, which require the space to have a certain richness in points.

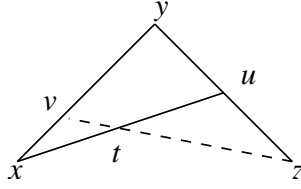


Figure 8. *Pasch's property.*

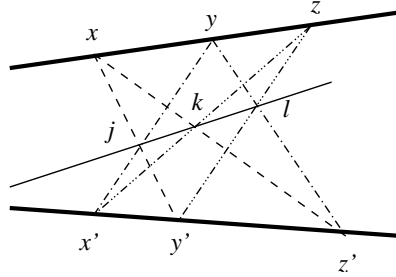


Figure 9. *Pappus property.*

The latter express typical geometrical behavior, witness *Pasch's axiom A7* (see Figure 8) and the earlier Pappus property (see Figure 9):

$$\forall x x' y y' z z' \exists j k l ((\beta(x y z) \wedge \beta(x' y' z') \wedge \beta(x j y') \wedge \beta(y j x') \wedge \beta(x k z') \wedge \beta(z k x') \wedge \beta(y l z') \wedge \beta(z l y')) \rightarrow \beta(j k l))$$

Moving to the opposite extreme of geometrical structure, consider the one-dimensional real line \mathbb{R} . Its universal first-order theory includes the strong dimension law

$$\forall x y z, \beta(x y z) \vee \beta(y x z) \vee \beta(x z y) \quad (10)$$

3.3. Modal languages of betweenness

Let us now turn to modal logic over affine spaces.

3.3.1. The basic language

Ternary betweenness models a binary betweenness modality $\langle B \rangle$:

$$M, x \models \langle B \rangle(\varphi, \psi) \quad \text{iff} \quad \exists y, z : \beta(y x z) \wedge M, y \models \varphi \wedge M, z \models \psi$$

Notice that this is a more standard modal notion than the earlier topological modality: we are working on frames, and there are no two-step quantifiers hidden in the

semantics. $\langle B \rangle$ is quite expressive. For instance, it defines one-step convex closure as follows:

$$\text{convex}(\varphi) \text{ iff } \langle B \rangle(\varphi, \varphi) \quad (11)$$

Passing to points ‘in between’ two others yields the convex closure only after repeated applications of this operator, as shown in Figure 15. In a more elaborate set-up, we could take a leaf from dynamic logic, and add an operation of *Kleene iteration* of the betweenness predicate—much as ternary ‘composition’ is iterated in dynamic Arrow Logic (cf. Chapter 8 in [BEN 96]). Next, the existential modality has an obvious dual universal version: $[B]\varphi \leftrightarrow \neg \langle B \rangle \neg \varphi \neg \psi$, which works out to

$$M, x \models [B](\varphi, \psi) \quad \text{iff} \quad \forall y, z : \beta(yxz) \rightarrow M, y \models \varphi \vee M, z \models \psi$$

An implicational variant of this definition is also helpful sometimes:

$$M, x \models [B](\neg \varphi, \psi) \quad \text{iff} \quad \forall y, z : \beta(yxz) \wedge M, y \models \varphi \rightarrow M, z \models \psi$$

One might think that there should be an independent conjunctive variant, saying that both endpoints have their property. But this is already definable—another sign of the strength of the language:

$$[B](\varphi, \perp) \wedge [B](\perp, \psi)$$

3.3.2. Versatile extensions

Betweenness is natural, but biased toward ‘interior positions’ of a segment. But given two points x and y , one can also consider all points z such that x lies in between y and z , or all w such that y lies in between x and w . In this way, two points identify a direction and a weak notion of orientation. There are two obvious further existential modalities corresponding to this. Together with $\langle B \rangle$, they form a ‘versatile’ triple in the sense of [VEN 92]. Such triples are often easier to axiomatize together than in isolation. As an illustration, consider the table of Figure 10, which we have been

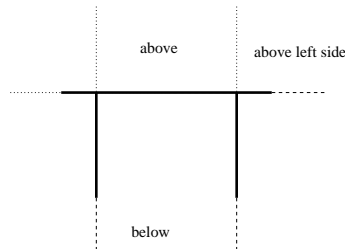


Figure 10. A table and the regions for versatile betweenness modalities.

setting in earlier sections. Using versatile modalities, the legs of the table and its top identify important zones of visual scenes, which also have names in natural language, such as everything ‘above the table’.

3.3.3. Affine transformations

Affine transformations are the invariant maps for affine geometry. Their modal counterpart are *affine bisimulations* which are mappings relating points verifying the same proposition letters, and maintaining the betweenness relation. We only display the definition for our original ‘interior’ betweenness—since the versatile extensions are straightforward:

Definition 3 (affine bisimulation) Given two affine models $\langle X, O, \beta, \nu \rangle$, and $\langle X', O', \beta', \nu' \rangle$, an *affine bisimulation* is a non-empty relation $\simeq \subseteq X \times X'$ such that, if $x \simeq x'$:

- (i) x and x' satisfy the same propositional letters,
- (ii) (*forth condition*): $\beta(yxz) \Rightarrow \exists y'z' : \beta'(y'x'z')$ and $y \simeq y'$ and $z \simeq z'$
- (iii) (*back condition*): $\beta'(y'x'z') \Rightarrow \exists yz : \beta(yxz)$ and $y \simeq y'$ and $z \simeq z'$

where $x, y, z \in X$ and $x', y', z' \in X'$.

In [GOL 87], isomorphisms are considered the only interesting maps across affine models. But in fact, just as with topological bisimulations versus homeomorphisms (Theorem 4), affine bisimulations are interesting coarser ways of comparing spatial situations. In the true modal spirit, they only consider the behavior of points inside their local line environments. Consider the two models consisting of 6 and 4 points, respectively, on and inside two triangles, with some atomic properties indicated, Figure 11. The models are evidently not isomorphic, but there is an affine bisimulation.

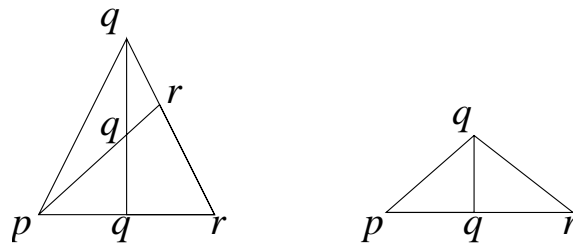


Figure 11. *Affine bisimilar models.*

Simply relate the two r points on the left with the single r point on the right. Then relate the top q point on the left with the top one on the right, the remaining two q points on the left with the one on the right, and, finally, the p point on the left with the one of the right. This affine bisimulation can be regarded as a sort of ‘modal contraction’ to a smallest bisimilar model. The models in Figure 12 are not bisimilar though. One can check that no relation does the job—or, more simply, note that the modal formula $q \wedge \langle B \rangle(r, r)$ holds on the q point of the left model and nowhere on the right. Affine bisimulations preserve truth of modal formulas in an obvious way, and hence they

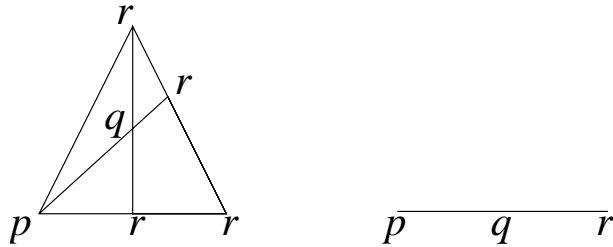


Figure 12. *Affine bisimilar reduction.*

are a coarser map than isomorphisms still giving meaningful geometrical invariances. This is exactly as we found with topological bisimulations versus homeomorphisms.

Incidentally, notice that there *is* a smaller bisimulation contraction for the left-hand triangle. The reason is that not all its points are uniquely definable in our modal language. The p and q points are uniquely definable, but all r points on the boundary satisfy the same modal statements. The contraction will look like the picture to the right, but with the middle point ‘in between’ the right point and the right point itself. (This is not a standard 2D ‘picture’, and duplicating points cannot always be contracted if we insist on those.) This situation would change with a modality for *proper betweenness*. Then the two middle r points become uniquely distinguishable as being properly in between different pairs of points. But the top and right-bottom point remain indistinguishable, unless we add versatile operators. It is a nice exercise to show that the triangle does have every point uniquely definable in the original language when we change the atomic proposition in the top vertex and the one center bottom to q and that in the middle of the right edge to p . Consider the new valuation in

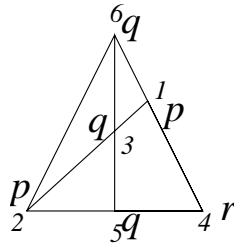


Figure 13. *An irreducible affine model.*

Figure 13. In this case there does *not* exist a bisimilar contraction. Every point of the triangle is distinguishable by a formula which is not true on any other point, see Figure 14. This suggests a theory of unique patterns, depending on how points are labeled in geometrical pictures. Finally, the geometrical setting suggests even *two* notions of

Point	Formula
1	$\varphi_1 = p \wedge \langle B \rangle(q, r)$
2	$\varphi_2 = p \wedge \neg\varphi_1$
3	$\varphi_3 = q \wedge \langle B \rangle(\varphi_1, \varphi_2)$
4	$\varphi_4 = r$
5	$\varphi_5 = q \wedge \langle B \rangle(\varphi_2, \varphi_4)$
6	$\varphi_6 = q \wedge \neg\varphi_3 \wedge \neg\varphi_5$

Figure 14. Formulas true at points of the model in Figure 13.

contraction. Bisimulations as defined above give no guarantee that the contraction will be a *planar* figure again. If we impose the latter extra requirement, standard results and algorithms change.

3.4. Modal logics of betweenness

The preceding language has a minimal logic as usual, which does not yet have much geometrical content. Its key axioms are two distribution laws:

$$\begin{aligned} \langle B \rangle(\varphi_1 \vee \varphi_2, \psi) &\leftrightarrow \langle B \rangle(\varphi_1, \psi) \vee \langle B \rangle(\varphi_2, \psi) \\ \langle B \rangle(\psi, \varphi_1 \vee \varphi_2) &\leftrightarrow \langle B \rangle(\psi, \varphi_1) \vee \langle B \rangle(\psi, \varphi_2) \end{aligned}$$

This minimal logic by itself has all the usual modal properties, decidability among them. Other basic principles express basic universal relational conditions, such as betweenness being symmetric in end-points, and all points lying ‘in between themselves’:

$$\begin{aligned} \langle B \rangle(\varphi, \psi) &\rightarrow \langle B \rangle(\psi, \varphi) \\ \varphi &\rightarrow \langle B \rangle(\varphi, \varphi) \end{aligned}$$

These facts are simple *frame correspondences* in the usual modal sense. A slightly more tricky example is the earlier-mentioned relational condition $\forall x \forall y \beta(xxy)$. This is not definable as it stands, but the modal axiom

$$(\varphi \wedge \langle B \rangle(\top, \psi)) \rightarrow \langle B \rangle(\varphi, \psi)$$

corresponds to the related principle

$$\forall x \forall y \forall z : \beta(zxy) \rightarrow \beta(xxy)$$

More generally, special modal axioms may correspond to more complex properties of geometric interest. Here is a clear example. Consider *associativity* of the betweenness modality:

$$\langle B \rangle(\varphi, \langle B \rangle(\psi, \xi)) \leftrightarrow \langle B \rangle(\langle B \rangle(\varphi, \psi), \xi)$$

Fact 3 Associativity corresponds to the Pasch Axiom.

Proof We spell out the simple correspondence argument, to show how easy matches can be between modal axioms and geometrical laws. Consider the Pasch Axiom **A7** in Tarski's list (Figure 8). Suppose that

$$\forall txyz \exists v(\beta(xtu) \wedge \beta(yuz) \rightarrow \beta(xvy) \wedge \beta(ztv))$$

holds in a frame. Assume that a point t satisfies $\langle B \rangle(\varphi, \langle B \rangle(\psi, \xi))$. Then there exist points x, u with $\beta(xtu)$ such that $x \models \varphi$, $u \models \langle B \rangle(\psi, \xi)$, and hence also points y, z with $\beta(yuz)$ such that $y \models \psi$ and $z \models \xi$. Now by Pasch's Axiom, there must be a point v with $\beta(xvy)$ and $\beta(vtz)$. Now, $v \models \langle B \rangle(\varphi, \psi)$ and hence $t \models \langle B \rangle(\langle B \rangle(\varphi, \psi), \xi)$. The other direction is similar.

Conversely, assume that $\beta(xtu)$ and $\beta(yuz)$. Define a valuation on the space by setting $\nu(p) = \{x\}$, $\nu(q) = \{y\}$, and $\nu(r) = \{z\}$. Thus, $u \models \langle B \rangle(q, r)$ and

$$t \models \langle B \rangle(p, \langle B \rangle(q, r)).$$

By the validity of modal Associativity, then

$$t \models \langle B \rangle(\langle B \rangle(p, q), r)$$

So there must be points v, w with $\beta(vtw)$ such that $v \models \langle B \rangle(p, q)$ and $w \models r$. By the definition of ν , the latter means that $w=z$, the former that $\beta(xuy)$. So indeed, u is the required point. QED

The preceding correspondence may be *computed automatically*, as Associativity has 'Sahlqvist form'. Thus, more general substitution methods apply for finding geometrical correspondents: cf. [BLA 01].

3.5. Special logics

For the affine modal logic of special models, additional considerations may apply. One example is the real line \mathbb{R} , which was also conspicuous in the topological setting. This time, the task is easy, as one can take advantage of the binary ordering $<$, defining

$$M, x \models \langle B \rangle(\varphi, \psi) \quad \text{iff} \quad \exists y, z : M, y \models \varphi \wedge M, z \models \psi \wedge z \leq x \leq y$$

Given this notion, we can use shorthand for the modalities of temporal logic: Future and Past (here, both including the present).

$$F\varphi := \langle B \rangle(\text{true}, \varphi)$$

$$P\varphi := \langle B \rangle(\varphi, \text{true})$$

Conversely, on \mathbb{R} , these two unary modalities suffice for defining $\langle B \rangle$:

$$\langle B \rangle(\varphi, \psi) \leftrightarrow P\varphi \wedge F\psi$$

Thus, a complete and decidable axiomatization for our $\langle B \rangle$ -language can be found using the well-known tense logic of future and past on \mathbb{R} [SEG 70].

Special models also raise special issues. We have already seen the universal axiom 10 defining one-dimensionality. What would be good versions for higher dimensions? For some further information on this, we refer to [AIE 02a]. Also, we will address this issue once more in our next section.

3.6. Logics of convexity

A binary modality for a ternary frame relation gives maximal flexibility. Nevertheless, given the geometrical importance of convexity per se, here is a unary modal operator for one-step convex closure:

$$M, x \models C\varphi \quad \text{iff} \quad \exists y, z : M, y \models \varphi \wedge M, z \models \varphi \wedge x \in y-z$$

This is a fragment of the preceding modal language:

$$C\varphi \leftrightarrow \langle B \rangle(\varphi, \varphi).$$

The axiomatic behavior is different though. In particular, distributivity fails. Of the axiom

$$C(\varphi \vee \psi) \leftrightarrow C\varphi \vee C\psi$$

only the right-to-left *monotonicity* implication is valid. But the one-step convex closure of a set of two distinct points is their whole interval, while the union of their separate one-step closures is just these points themselves.

Earlier on, we already noted that one-step convex closure needs finite iteration to yield the usual convex closure of geometry. This could be brought out again in a language with an additional modality C^* , where the $*$ denotes Kleene iteration. This interesting spatial use of dynamic logic is not pursued here, for a reason to be explained below. First, note that the non-idempotence of C gives additional expressive power by itself. In fact, it helps us distinguish dimensions! Here is how. The principle

$$CC\varphi \leftrightarrow C\varphi$$

holds on \mathbb{R} , but not on \mathbb{R}^2 . A counter-example on \mathbb{R}^2 is shown in Figure 15. The region p is given by three non-collinear points. Cp is then the bare triangle: convexity has added the edges. Applying convexity again, CCp defines a different region, namely the whole triangle with its interior. One may be inclined to rush to the conclusion that principles of the form

$$C^{n+1}\varphi \leftrightarrow C^n\varphi \tag{12}$$

determine the dimensionality of the spaces \mathbb{R}^n for all n . But here is a surprise.

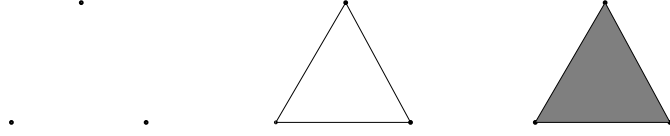


Figure 15. In a two dimensional space, the sequential application of the convexity operator to three non aligned points results in two different regions: a triangle (only the sides and corners of it) and the filled triangle.

Theorem 9 The principle $CCC\varphi \leftrightarrow CC\varphi$ holds in \mathbb{R}^3 .

Proof Here is a sketch. It will help the reader to visualize the situation using the tetrahedron example in Figure 17.

$C\varphi$ consists of all points in between two φ -points. $CC\varphi$ consists of all points in between the latter, and the implication $CC\varphi \rightarrow C\varphi$ corresponds (in the literal modal frame-theoretic sense) to the betweenness property that

$$(\beta(yxz) \wedge \beta(uyv) \wedge \beta(szt) \rightarrow \bigwedge \{\beta(ixj) \mid i, j \in \{u, v, s, t\}\})$$

This is true in one dimension, though not in higher ones.

On the plane, $C\varphi$ consists of the same points. But we can give another description of $CC\varphi$. If x lies in between two $C\varphi$ -points, say on intervals $y-z$ and $u-v$, respectively, then x lies in/on one of the triangles yzu or yzv . Therefore, $CC\varphi$ -points lie on triangles of φ -points. Now consider any point r in $CCC\varphi$, i.e., between points s, t in/on such $CC\varphi$ triangles. Intersecting the segment $s-t$ with the two triangle boundaries, we get that r lies in a four sided polygon of φ -points, and hence, bisecting, r is already in/on a triangle of φ -points: i.e., r is in $CC\varphi$ already.

In 3D, the description for $CC\varphi$ is different, because the two segments for the $C\varphi$ -points need not lie in the same plane. The outcome is that these points lie in/on a 4-hedron of φ -points. Now consider a generic point r in $CCC\varphi$. It will lie in between points in such 4-hedra. This situation is easier to picture: take the segment on which it lies, and intersect that with the relevant faces of the 4-hedra. Then it is easy to see that the point r lies inside a 6-hedron whose vertexes are φ -points. But then, cutting this up a number of times now, there is again a 4-hedron of φ -points in/on which we find r , hence, it is in $CC\varphi$ already. QED

In [AIE 02a], we prove the above by a matrix representation on the projective plane. (We think that proof generalizes to higher dimensions.) As a corollary, for real spaces, we can then define convex closure in our language after all, using CC combinations. Hence a full dynamic language, no matter how interesting per se, is not strictly needed. But for the moment we just notice the following fact.

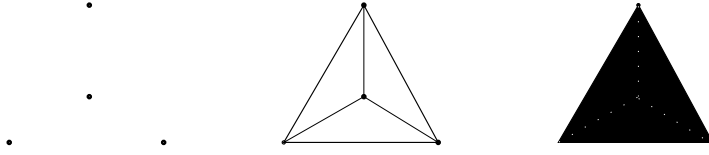


Figure 16. In a three dimensional space, the sequential application of the betweenness operator to four non coplanar points results in two distinct regions: the wire-frame of a tetrahedron and the filled tetrahedron.

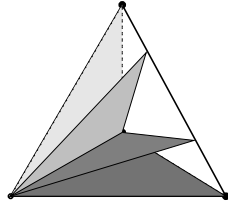


Figure 17. Applying convexity from the wire-frame to the full tetrahedron.

Fact 4 C^n is a convex set in \mathbb{R}^n .

Now there are dimension highlighters in our language after all. An old theorem from almost a century ago [HEL 23] comes to the rescue:

Theorem 10 (Helly) If K_1, K_2, \dots, K_m are convex sets in n -dimensional Euclidean space E^n , in which $m > n + 1$, and if for every choice of $n + 1$ of the sets K_i there exists a point that belongs to all the chosen sets, then there exists a point that belongs to all the sets K_1, K_2, \dots, K_m .

This theorem does have a modal version;

$$\bigwedge_{f:\{1,\dots,n+1\}\rightarrow\{1,\dots,m\}} E(\bigwedge_{i=1}^{n+1} (C^n \varphi_{f(i)}) \rightarrow E(\bigwedge_{i=1}^m C^n \varphi_i)$$

where E is the existential modality (defined in terms of betweenness in Equation 9), C^n is convexity applied n times (Fact 4), and f is a function from $\{1, \dots, n + 1\}$ to $\{1, \dots, m\}$.

3.7. First-order affine geometry

The above modal language is again a fragment of a first-order one, under the standard translation. The relevant first-order language is not quite that of Tarski's ele-

mentary geometry for \mathbb{R}^2 , as we also get unary predicate letters denoting regions. In fact, one open question which we have not been able to resolve is this. A formula $\varphi(\beta, P, Q, \dots)$ is valid, say in the real plane, if it holds for any interpretation of the regions P, Q, \dots . Thus, we would be looking at a universal fragment of a *monadic second-order logic*:

What is the complete monadic Π_1^1 theory of the affine real plane?

We suspect it is recursively axiomatizable and decidable—perhaps using the Ehrenfeucht game methods of [DOE 87]. This is an extension of the affine part of Tarski’s logic. But our previous discussion has also identified interesting *fragments*:

What is the universal first-order theory of the affine real plane?

As in our discussion of topology, the affine first-order language of regions is a natural limit toward which modal affine languages can strive via various logical extensions. From a geometrical viewpoint, one might also hope that ‘layering’ the usual language in this modal way will bring to light interesting new geometrical facts.

Another major feature of standard geometry is the *equal status of points and lines*. This would suggest a reorganization of the modal logic to a *two-sorted* one stating properties of both points and segments, viewed as independent semantic objects. There are several ways of doing this. One would be a *two-dimensional* modal language in the spirit of [MAR 97], handling both points and pairs of points, with various cross-sortal modalities. Another would treat both objects as primitives, and then have cross-sortal modalities for “at an endpoint,” “at an intermediate point,” “at some surrounding segment.” We think the latter is the best way to go eventually, as it has the useful feature of replacing talk in terms of ternary relations, which are hard to visualize, by binary ones, which are easier to represent. (This is of course the key advantage of the geometer’s habit of working with points and lines.) Moreover, the two-sorted move would be in line with other modal trends such as *Arrow Logic* [BEN 96, VEN 96], where transitions between points become semantical objects in their own right. This gives more control over semantic structures and the complexity of reasoning. It would also help reflect geometrical duality principles of the sort that led from affine to projective geometry.

4. Metric geometry

There is more structure to geometry than just affine point-line patterns. Tarski’s equidistance also captures *metric* information. There are various primitives for this. Tarski used quaternary equidistance—while ternary equidistance would do just as well (x, y and z lie at equal distances). Our choice in this Section is a different one, stressing the comparative character of metric structure.

4.1. The geometry of relative nearness

Relative nearness was introduced in [BEN 83b] (see Figure 18):

$$N(x, y, z) \text{ iff } y \text{ is closer to } x \text{ than } z \text{ is, i.e., } d(x, y) < d(x, z)$$

where $d(x, y)$ is any distance function.

This is meant very generally. The function d can be a geometrical metric, or some

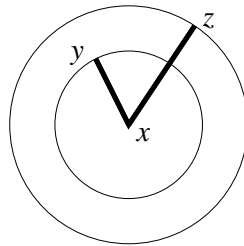


Figure 18. From point x , y is closer than point z .

more cognitive notion of visual closeness (van Benthem's original interest; cf. also Gärdenfors 'Conceptual Spaces'), or some utility function with metric behavior. In [RAN 01], Randell et al. develop the theory of comparative nearness for the purpose of robot navigation, related to the earlier-mentioned calculus of regions RCC.

Relative nearness defines equidistance:

$$Eqd(x, y, z) : \neg N(x, y, z) \wedge \neg N(x, z, y)$$

Tarski's quaternary equidistance is expressible in terms of N as well. Details are postponed until Section 4.3 on first-order metric geometry.

Affine betweenness is also definable in terms of N , at least in real spaces \mathbb{R}^n : cf. Section 4.2. Finally, note that even identity of points $x = y$ is expressible in terms of N

$$x = y \text{ iff } \neg N(x, x, y)$$

At the end of the XVIII century the mathematician Lorenzo Mascheroni proved in his tractate *The Geometry of Compasses* that everything that can be done with compass and ruler can be done with the compass alone. One can generate all of Mascheroni's constructions with the first-order logic of N and thereby achieve geometry. Examples are presented in [AIE 02a].

The further analysis of this structure can proceed along much the same lines as the earlier one for affine geometry. In particular, as a source of basic constraints,

one is interested in the *universal first-order theory* of relative nearness. Its complete description is an open question right now, but here are some examples showing its interest. First, comparative nearness induces a standard comparative ordering. Once a point x is fixed, the binary order $N(x, y, z)$ is irreflexive, transitive and almost-connected:

$$\forall x \forall y \forall z \forall u : (N(x, y, z) \wedge N(x, z, u)) \rightarrow N(x, y, u) \quad (\text{transitivity})$$

$$\forall x \forall y : \neg N(x, y, y) \quad (\text{irreflexivity})$$

$$\forall x \forall y \forall z \forall u : N(x, y, z) \rightarrow (N(x, y, u) \vee N(x, u, z)) \quad (\text{almost-connectedness})$$

These are like the principles of comparative order in logical semantics for counterfactuals [LEW 73]. But additional valid principles are more truly geometrical, relating distances from different standpoints. These are the following *triangle inequalities*

$$\forall x \forall y \forall z \forall u : N(x, y, z) \wedge N(z, x, y) \rightarrow N(y, x, z)$$

$$\forall x \forall y \forall z \forall u : \neg N(x, y, z) \wedge \neg N(z, x, y) \rightarrow \neg N(y, x, z)$$

These seem pretty universal constraints on comparative nearness in general. Further universal first-order properties of N reflect the two-dimensionality of the plane. Just inscribe 6 equilateral triangles in a circle, and see that

on a circle with radius r , the largest polygon that can be inscribed of points at distance r has 6 vertexes.

This upper bound can be expressed in universal first-order form, because we can express equidistance in terms of N . Other principles of this form concern the arrangement of points on circles:

on a circle C , any point has at most two points at each of its ‘equidistance levels’ on C

and

circles with different centers intersect in at most two points.

To obtain the complete universal first-order theory of comparative nearness in the Euclidean plane \mathbb{R}^2 , one would have to guarantee a planar embedding. Do our general axioms, including the triangle inequalities, suffice for axiomatizing the complete universal theory of *all* Euclidean spaces \mathbb{R}^n ?

4.2. Modal logic of nearness

The ternary relation of comparative nearness lends itself to modal description, just like ternary betweenness. We will just briefly sketch the resulting logic, which is like our affine system in its broad outline. But the intuitive meaning of N also adds some new issues of its own.

4.2.1. Modal languages of nearness

First, one sets

$$M, x \models \langle N \rangle \phi, \psi \text{ iff } \exists y, z : M, y \models \psi \wedge M, z \models \phi \wedge N(x, y, z)$$

The universal dual is also interesting in its spatial behavior:

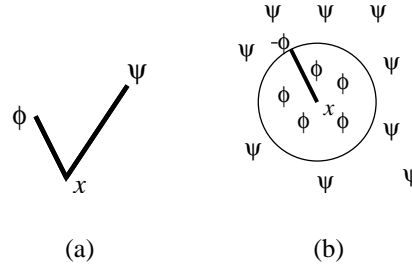


Figure 19. Interpreting a modal operator of nearness and its dual.

$$M, x \models [N] \phi, \psi \text{ iff } \forall y, z : N(x, y, z) \wedge M, y \models \neg \phi \rightarrow M, z \models \psi$$

Dropping the negation, one gets an interchangeable version with the following intuitive content:

if any point y around the current point x satisfies ϕ , then all points z further out must satisfy ψ .

Moreover, there are obvious *versatile* versions of these modal operators, which look at the same situation in a different way. For instance, using one of these in its universal version, we can also express the appealing statement that

if any point y around the current point x satisfies ϕ , then all points z closer to x must satisfy ψ .

See Figure 20 for an illustration. Finally, note that this language defines an existential modality (assuming the mild condition that $\forall y : N(x, x, y) \vee x = y$):

$$E\phi \text{ iff } \phi \vee \langle N \rangle (\top, \phi)$$

Without the condition, this modality will only range over connected components.

4.2.2. Modal logics of nearness

Modal logics of nearness again start with universally valid principles, with distribution as the prime example:

$$\langle N \rangle (\phi \vee \psi, \xi) \leftrightarrow \langle N \rangle (\phi, \xi) \vee \langle N \rangle (\psi, \xi)$$

$$\langle N \rangle (\phi, \psi \vee \xi) \leftrightarrow \langle N \rangle (\phi, \psi) \vee \langle N \rangle (\phi, \xi)$$

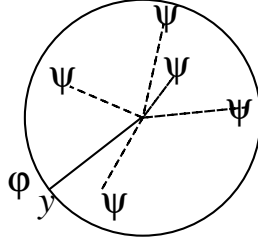


Figure 20. Versatile interpretations of the dual of a modal operator of nearness.

Universal constraints of earlier kinds return as special axioms. Here is an example:

$$\langle N \rangle(\varphi, \psi) \wedge \neg \langle N \rangle(\varphi, \varphi) \wedge \neg \langle N \rangle(\psi, \psi) \wedge \langle N \rangle(\psi, \xi) \rightarrow \langle N \rangle(\varphi, \xi) \quad (\text{transitivity})$$

In the above definition the two clauses $\neg \langle N \rangle(\varphi, \varphi)$ and $\neg \langle N \rangle(\psi, \psi)$ are necessary to ensure that φ and ψ can be true only at a fixed distance from the current point. Omitting them results in an invalid principle, as it may very well be the case that $\langle N \rangle(\varphi, \psi) \wedge \langle N \rangle(\psi, \xi) \wedge \neg \langle N \rangle(\xi, \varphi)$ if φ is true at points at different distances from the current one. Another example of a universal constraint is almost-connectedness:

$$\langle N \rangle(\varphi, \psi) \wedge \neg \langle N \rangle(\varphi, \varphi) \wedge \neg \langle N \rangle(\psi, \psi) \wedge E\xi \rightarrow \langle N \rangle(\varphi, \xi) \vee \langle N \rangle(\xi, \psi) \quad (\text{almost-connectedness})$$

Irreflexivity seems harder to define (as usual in modal logics), but see below.

Special logics of nearness arise by looking at special structures, or at least, imposing more particular constraints. These can again be computed by correspondence techniques. In a similar way, one can modally express the *triangle inequalities*. But in fact, there is a more general observation to be made here. Note that our language can define that φ holds in a unique point:

$$E! \varphi \text{ iff } E(\varphi \wedge \neg \langle N \rangle(\varphi, \varphi))$$

Now observe the following.

Proposition 11 *Every universal first-order property of N is modally definable.*

Proof Every such property is of the form

$$\forall x_1 \dots \forall x_k : BC(N(x_i, x_j, x_k))$$

where BC stands for any boolean combination of nearness atoms. Now take proposition letters p_1, \dots, p_k and write

$$E! p_1 \wedge \dots \wedge E! p_k \rightarrow BC(N^\#(x_i, x_j, x_k))$$

where $N^\#(x_i, x_j, x_k)$ is defined as $E(p_i \wedge \langle N \rangle(p_j, p_k))$. It is evident that this is a modal frame correspondent. QED

The technique in this proof is well-known from extended modal logics with nominals or difference modalities, which are complete for defining universal frame conditions. This also explains the definition of the triangle inequalities. Moreover, *irreflexivity* (whose first-order definition is $\forall x \forall y \neg N(x, y, y)$) is definable after all by

$$E!p_1 \wedge E!p_2 \rightarrow \neg E(p_1 \wedge \langle N \rangle(p_2, p_2))$$

4.2.3. Modal extensions

Useful modal extensions of the base language are partly as in the affine case. But there is also a novelty. In describing spatial patterns, one may often want to say something like this:

for every φ -point around x , *there exists* some closer ψ -point.

Now this is not definable in our language, which uses uniform *EE* or *AA* quantifier combinations. Mixing universal and existential quantifiers is more like temporal ‘Until’ languages. Speaking generally, we want a new operator:

$$M, x \models \langle N^{\exists\forall} \rangle(\varphi, \psi) \text{ iff } \forall y (M, y \models \varphi \rightarrow \exists z (N(z, y, x) \wedge M, z \models \psi))$$

The general logic of this additional modality over a ternary relation is a bit more complex with respect to distribution and monotonicity behavior—but it can be axiomatized completely, at least minimally, over all abstract models.

Indeed, this universal-existential pattern is reminiscent of other modal logics naturally involving ternary frame relations. One example is *temporal logic* of SINCE and UNTIL, which involves moving to some point around the current point in time, and then saying something about all points in between. One existential-universal variant of the preceding modality would indeed be a kind of spatial UNTIL, stating that some point on a circle around the current point satisfies φ , while all points in the interior satisfy ψ . This is almost a metric analogue of the topological UNTIL operator in Section 2.2.2, but the latter should have the whole circle boundary satisfy φ , which requires one more universal modality over equidistant points.

Another intriguing analogy is with a typical modal logic over comparative nearness, viz. *conditional logic*. The latter is mostly known in connection with counterfactuals and default reasoning [LEW 73, NUT 83, VEL 85]. In general conditional logic, one crucial binary modality reads

$$\varphi \Rightarrow \psi \text{ iff every closest } \varphi\text{-world is } \psi$$

This satisfies the usual Lewis axioms for conditional semantics in terms of ‘nested spheres’ (cf. [BEN 83a]):

$$\begin{aligned}
& \varphi \Rightarrow \psi \rightarrow \varphi \Rightarrow \psi \vee \xi \\
& \varphi \Rightarrow \psi \wedge \varphi \Rightarrow \xi \rightarrow \varphi \Rightarrow \psi \wedge \xi \\
& \varphi \Rightarrow \psi \wedge \varphi \Rightarrow \xi \rightarrow \varphi \wedge \psi \Rightarrow \xi \\
& \varphi \Rightarrow \psi \wedge \xi \Rightarrow \psi \rightarrow \varphi \vee \xi \Rightarrow \psi \\
& ((\varphi \vee \psi) \Rightarrow \varphi) \vee (\neg((\varphi \vee \psi) \Rightarrow \xi)) \vee (\psi \Rightarrow \xi)
\end{aligned}$$

The interesting open question here concern modal-conditional reflections of the additional geometrical content of the $N(x, y, z)$ relation. Lewis' complete system still concerns just ordering properties of comparisons from some fixed vantage point. This shows in the fact that there are no significant axioms for *iterated conditionals* which require shifts in vantage point. What is the conditional logic content of the triangle inequalities?

4.3. First-order theory of nearness

As for the complete first-order theory of relative nearness, we have no special results to offer, except for the promised proof of an earlier claim.

Fact 5 The single primitive of comparative nearness defines the two primitives of Tarski's Elementary Geometry in first order logic.

Proof The following defines betweenness (see Figure 21):

$$\beta(yxz) \text{ iff } \neg \exists x' : N(y, x', x) \wedge N(z, x', x)$$

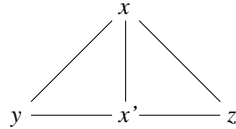


Figure 21. Defining betweenness via nearness.

This allows us to define parallel segments in the usual way, as having no intersection points on their generated lines.

$$\begin{aligned}
xx' || yy' & \leftrightarrow \neg \exists c : \beta(xx'c) \wedge \beta(yy'c) \wedge \\
& \neg \exists c' : \beta(c'xx') \wedge \beta(cyy') \wedge \\
& \neg \exists c'' : \beta(cxx'') \wedge \beta(cyy'')
\end{aligned}$$

Then, one defines equal segment length by

$$\delta(x, y, z, u) \text{ iff } \exists y' : xu || yy' \wedge xy || uy' \wedge \neg N(u, z, y') \wedge \neg N(u, y'z)$$

Intuitively, one moves one segment on the other matching end-points and preserving length via parallel lines. Then state that the other end-points are at the same distance from the joined point. See the depiction in Figure 22. We leave suitable formulations for higher-dimensional real spaces to the reader. QED

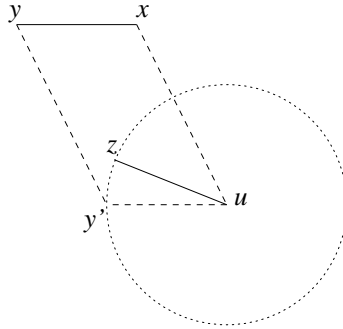


Figure 22. *Equidistance in terms of nearness.*

Apart from this, much of our earlier discussion concerning affine first-order geometry applies. Incidentally, no claim is made here for the originality of this approach per se. There are many systems of logical geometry which have similar richness. A case in point is the axiomatization of constructive geometry in [PLA 95].

5. Linear algebra

Our final example of modal structures inside a spatial theory is different in spirit from either topology or standard geometry. Connections between *linear algebra* and spatial representation are well-known from a major qualitative visual theory, viz. *mathematical morphology*. Our treatment follows the lines of [BEN]. (A different connection between mathematical morphology and modal logic is found in [BLO 00], which also includes a fuzzy version.) The flavor of this brand of spatial reasoning is different from what we had before—but similar modal themes emerge all the same.

Mathematical morphology, developed in the 60s by Matheron and Serra, [MAT 67, SER 82], underlies modern image processing. where it has a wide variety of applications. Compared with classical signal processing approaches it is more efficient in image preprocessing, enhancing object structure, and segmenting objects from the background. The modern mathematics behind this involves lattice theory: [HEI 94]. Logicians may want to think of ‘linear algebras’ [GIR 87], an abstract version of vector spaces:

Definition 4 (linear algebra) $\langle X, \sqcap, \sqcup, \perp, \neg, \star, \check{0}, \check{1} \rangle$ is a *linear algebra* if

- (i) $\langle X, \sqcap, \sqcup, \perp, \rangle$ is a lattice with bottom \perp ;
- (ii) $\langle X, \star, \check{1} \rangle$ is a monoid with unit $\check{1}$;
- (iii) if $x \leq x', y \leq y'$, then $x \star y \leq x' \star y'$ and $x' \dashv\circ y \leq x \dashv\circ y'$;
- (iv) $x \star y \leq z$ iff $x \leq y \dashv\circ z$;
- (v) $x = (x \dashv\circ \check{0}) \dashv\circ \check{0}$ for all x .

In line with our spatial emphasis of this paper, we will stick with concrete vector spaces \mathbb{R}^n in what follows. Images are regions consisting of sets of vectors. Mathematical morphology provides four basic ways of combining, or simplifying images, viz. *dilation*, *erosion*, *opening* and *closing*. These are illustrated pictorially in Figure 23. Intuitively, dilation adds regions together—while, e.g., erosion is a way of

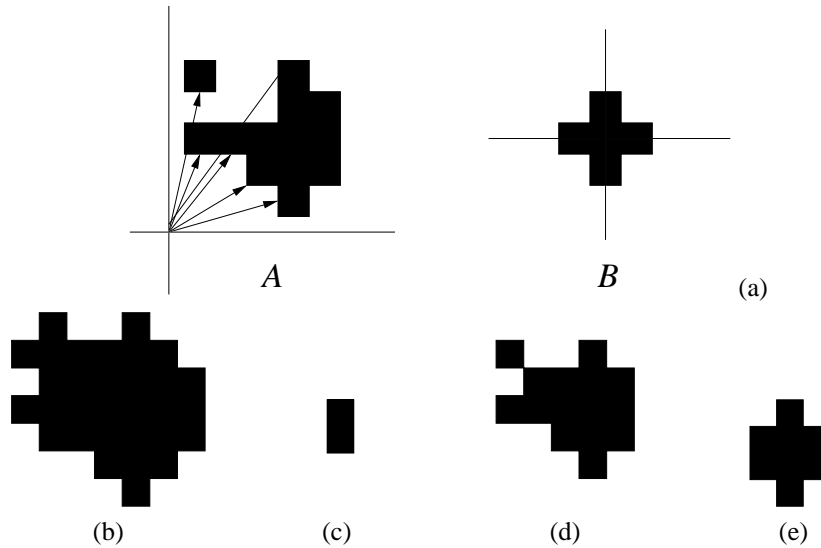


Figure 23. (a) Regions A and B , elements of the vector space \mathbb{N}^2 ; (b) dilating A by B ; (c) eroding A by B ; (d) closing A by B ; (e) opening A by B .

removing ‘measuring idiosyncrasies’ from a region A by using region B as a kind of boundary smoothener. (If B is a circle, one can think of it as rolling tightly along the inside of A ’s boundary, leaving only a smoother interior version of A .) More formally, dilation, or *Minkowski addition* \oplus is vector sum:

$$A \oplus B = \{a + b | a \in A, b \in B\} \quad \text{dilation}$$

This is naturally accompanied by

$$A \ominus B = \{a | a + b \in A, \forall b \in B\} \quad \text{erosion}$$

Openings and closing are combinations of dilation and erosions:

the structural *opening* of A by B $(A \ominus B) \oplus B$
the structural *closing* of A by B $(A \oplus B) \ominus B$

In addition, mathematical morphology also employs the usual Boolean operations on regions: intersection, union, and complement. This is our third mathematization of real numbers \mathbb{R}^n in various dimensions, this time focusing on their vector structure. Evidently, the above operations are only a small sub-calculus, chosen for its computational utility and expressive perspicuity.

5.1. Mathematical morphology and linear logic

The first connection that we note lies even below the level of standard modal languages. The Minkowski operations behave a bit like the operations of *propositional logic*. Dilation is like a logical conjunction \oplus , and erosion like an implication \longrightarrow , as seems clear from their definitions ('combining an A and a B ', and 'if you give me a B , I will give an A '). The two were related by the following *residuation law*:

$$A \bullet B \subseteq C \text{ iff } A \subseteq B \longrightarrow C$$

which is also typical for conjunction and implication (cf. also clause (iv) in Definition 4). Thus, \longrightarrow is a sort of inverse to \oplus .

5.1.1. Resource logics

There already exists a logical calculus for these operations, invented under the *multiplicative linear logic* name in theoretical computer science [TRO 92], and independently as the *Lambek calculus with permutation* in logical linguistics, cf. [KUR 95]. The calculus derives 'sequents' of the form $A_1, \dots, A_k \Rightarrow B$ where each expression A, B in the current setting stands for a region, and the intended interpretation—in our case—says that

the sum of the A 's is included in the region denoted by B .

Here are the derivation rules, starting from basic axioms $A \Rightarrow A$:

$$\frac{X \Rightarrow A \quad Y \Rightarrow B}{X, Y \Rightarrow A \bullet B} \qquad \frac{X, A, B \Rightarrow C}{X, A \bullet B \Rightarrow C} \quad (\text{product rules})$$

$$\frac{A, X \Rightarrow B}{X \Rightarrow A \longrightarrow B} \qquad \frac{X \Rightarrow A \quad B, Y \Rightarrow C}{X, A \longrightarrow B, Y \Rightarrow C} \quad (\text{arrow rules})$$

$$\frac{X \Rightarrow A}{\pi[X] \Rightarrow A} \text{permutation} \qquad \frac{X \Rightarrow A \quad A, Y \Rightarrow B}{X, Y \Rightarrow B} \text{cut} \quad (\text{structural rules})$$

Derivable sequents typically include:

$$A, A \longrightarrow B \Rightarrow B \quad (\text{'function application'})$$

$$A \longrightarrow B, B \longrightarrow C \Rightarrow A \longrightarrow C \quad (\text{'function composition'})$$

Here is an example of a derivation, just for the flavor of the system:

$$\frac{\frac{\frac{A \Rightarrow A \quad B \Rightarrow B}{A, A \longrightarrow B \Rightarrow B} \quad \overline{C \Rightarrow C}}{A, A \longrightarrow B, B \longrightarrow C \Rightarrow C}}{A \longrightarrow B, B \longrightarrow C \Rightarrow A \longrightarrow C}$$

Another key example are the two ‘Currying’ laws, whose proof uses the \bullet rules:

$$(A \bullet B) \longrightarrow C \Rightarrow (A \longrightarrow (B \longrightarrow C))$$

$$(A \longrightarrow (B \longrightarrow C)) \Rightarrow (A \bullet B) \longrightarrow C$$

This calculus is best understood in terms of *resources*. Think of each premise in an argument as a resource which you can use just once when ‘drawing’ the conclusion. In standard logical inference, the premises form a set: you can duplicate the same item, or contract different occurrence of it without any change in valid conclusions. This time, however, the premises form a *bag*, or *multi-set*, of occurrences: validating only ‘resource-conscious’ versions of the standard logical laws. E.g., ‘Modus Ponens’ $A, A \longrightarrow B \Rightarrow B$ is valid, but its variant $A, A, A \longrightarrow B \Rightarrow B$ is not: having one unused resource left. A correct provable sequent using the latter resources is rather:

$$A, A, A \longrightarrow B \Rightarrow A \bullet B$$

Or consider the classically valid sequent $A, (A \longrightarrow (A \longrightarrow B)) \Rightarrow B$. Here the above calculus only proves $A, (A \longrightarrow (A \longrightarrow B)) \Rightarrow A \longrightarrow B$, and you must supply one more resource A to derive

$$A, A, (A \longrightarrow (A \longrightarrow B)) \Rightarrow B.$$

The related categorial grammar interpretation for this same calculus reads the product \bullet as syntactic juxtaposition of linguistic expressions, and an implication $A \longrightarrow B$ as a function category taking A -type expressions to B -type expressions. The same occurrence-based character will hold: repeating the same word is not the same as having it once.

The major combinatorial properties of this calculus **LL** are known, including proof-theoretic cut elimination theorems, and *decidability* of derivability in NP time. Moreover, there are several formal semantics underpinning this calculus (algebraic, game-theoretic, category-theoretic, possible worlds-style [BEN 91a]). Still, no totally satisfying modeling has emerged so far.

5.1.2. Linear logic as mathematical morphology

Here is where the present setting becomes intriguing: mathematical morphology provides a new model for linear logic!

Fact 6 Every space \mathbb{R}^n with the Minkowski operations is a model for all **LL**-provable sequents.

This soundness theorem shows that every sequent one derives in **LL** must be a valid principle of mathematical morphology. One can see this for the above examples, or other ones, such as the idempotence of morphological opening $(A \ominus B) \oplus B$:

$$(((A \ominus B) \oplus B) \ominus B) \oplus B = ((A \ominus B) \oplus B)$$

In **LL**, the opening is $(A \longrightarrow B) \bullet A$, and the idempotence law is literally derivable using the above rules:

$$\begin{aligned} (A \longrightarrow B) \bullet A &\Rightarrow (A \longrightarrow ((A \longrightarrow B) \bullet A)) \bullet A \\ (A \longrightarrow ((A \longrightarrow B) \bullet A)) \bullet A &\Rightarrow (A \longrightarrow B) \bullet A \end{aligned}$$

The list might even include new principles not considered in that community. The converse seems an open completeness question of independent interest:

*Is multiplicative linear logic complete w.r.t the class of all \mathbb{R}^n 's?
Or even w.r.t. two-dimensional Euclidean space?*

Further, mathematical morphology laws ‘mix’ pure Minkowski operations \oplus, \longrightarrow with standard Boolean ones. E.g. they include the fact that $A \longrightarrow (B \cap C)$ is the same as $(A \cup B) \longrightarrow C = (A \longrightarrow C) \cap (B \longrightarrow C)$. This requires adding Boolean operations to **LL**:

$$\begin{array}{ccc} \frac{X, A \Rightarrow B}{X, A \cap C \Rightarrow B} & \frac{X, A, \Rightarrow B}{X, C \cap A \Rightarrow B} & \frac{X \Rightarrow A \quad X \Rightarrow B}{X \Rightarrow A \cap B} \\ \\ \frac{X \Rightarrow A}{X \Rightarrow A \cup B} & \frac{X \Rightarrow A}{X \Rightarrow B \cup A} & \frac{X, A \Rightarrow B \quad X, C \Rightarrow B}{X, A \cup C \Rightarrow B} \end{array}$$

Note the difference between the two conjunctions. Product \bullet and intersection have some similarities, but the rules are different. E.g., $A \longrightarrow (B \bullet C)$ does not derive $(A \longrightarrow B) \bullet (A \longrightarrow C)$, or vice versa. Conversely, dot product satisfied the ‘Curry laws’, but $(A \cap B) \longrightarrow C$ is certainly not derivably equivalent to $(A \longrightarrow (B \longrightarrow C))$. All these observations tally with known facts in mathematical morphology. Indeed, the extended calculus is still sound—while its completeness remains an open question.

The Boolean operations look a bit like the ‘additives’ of linear logic, but they also recall ordinary modal logic, which is where we are going now.

5.2. Richer languages

Evidently, the basic players in an algebra of regions in a vector space are the vectors themselves. For instance, Figure 23.a represents the region A as a set of 13 vectors departing from the origin. Vectors come with some natural operations, such as binary addition, or unary inverse—witness the usual definition of a vector space. A vector v in our particular spaces may be viewed as an ordered pair of points (o, e) , with o the origin and e the end point. Pictorially, this is an arrow from o to e . Now this provides our point of entry into modal logic.

5.2.1. Arrow logic

Arrow logic is a form of modal logic where the objects are transitions or arrows, structured by various relations. In particular, there is a binary modality for *composition* of arrows, and a unary one for *converse*. The motivation for this comes from dynamic logics, treating transitions as objects in their own right, and from relational algebra, making pairs of points separate objects. This allows for greater expressive power than the usual systems, while also lowering complexity of the core logics (see [BLA 01, BEN 96] for overviews). Consider in particular the pair-interpretation, with arrows being pairs of points (a_o, a_e) . Here are the fundamental semantic relations:

composition $C(a_o, a_e)(b_o, b_e)(c_o, c_e)$ iff $a_o = b_o$, $a_e = c_e$, and $b_e = c_o$,

inverse $R(a_o, a_e)(b_o, b_e)$ iff $a_o = b_e$, and $a_e = b_o$,

identity $I(a_o, a_e)$ iff $a_o = a_e$.

An abstract model is then defined as any set of arrows as primitive objects, with three relations as above, and a valuation function sending each proposition letter p to the set of the arrows where property p holds.

Definition 5 (arrow model) An *arrow model* is a tuple $M = \langle W, C, R, I, \nu \rangle$ such that $C \subseteq W \times W \times W$, $R \subseteq W \times W$, $I \subseteq W$, and $\nu : W \rightarrow P$.

Such models have a wide variety of interpretations, ranging from concrete models in linguistic syntax to abstract ones in category theory [VEN 96]—but of relevance to us is the obvious connection with vector spaces. Think of $Cxyz$ as $x = y + z$, Rxy as $x = -y$ and Ix as $x = 0$. To make this even clearer, we use a modal arrow language with proposition letters, the identity element 0 , monadic operators \neg , $-$, and a dyadic operator \oplus . The truth definition reads:

$M, x \models p$	iff	$a \in \nu(p)$
$M, x \models 0$	iff	Ix
$M, x \models \neg\varphi$	iff	$\exists y : Rxy$ and $M, y \models \varphi$
$M, x \models \neg\varphi$	iff	not $M, x \models \varphi$
$M, x \models \varphi \vee \psi$	iff	$M, x \models \varphi$ or $M, x \models \psi$
$M, x \models A \oplus B$	iff	$\exists y \exists z : Cxyz \wedge M, y \models A \wedge M, z \models B$
$M, x \models A \ominus B$	iff	$\forall y \forall z : Cxyz \wedge M, z \models A \rightarrow M, y \models B$

This system can be studied like any modal logic. For the basic results in the area, we refer to the above-mentioned publications.

5.2.2. Arrow logic as linear algebra

Most modal topics make immediate sense in linear algebra or mathematical morphology. E.g., the above models support a natural notion of *bisimulation*:

Definition 6 (arrow bisimulation) Let M, M' be two arrow models. A relation $\rightleftharpoons \subseteq W \times W'$ is an *arrow bisimulation* iff, for all x, x' such that $x \rightleftharpoons x'$:

base $x \in \nu(p)$ iff $x' \in \nu'(p)$,

C-forth $Cxyz$ only if there are $y'z' \in W'$ such that $C'x'y'z', y \rightleftharpoons y'$ and $z \rightleftharpoons z'$,

C-back $C'x'y'z'$ only if there are $yz \in W$ such that $Cxyz, y \rightleftharpoons y'$ and $z \rightleftharpoons z'$,

R-forth Rxy only if there are $y' \in W'$ such that $R'x'y'$ and $y \rightleftharpoons y'$,

R-back $R'x'y'$ only if there are $y \in W$ such that Rxy and $y \rightleftharpoons y'$,

I-harmony Ix iff $I'x'$.

Arrow bisimulation is a coarser comparison of vector spaces than the usual linear transformations. It preserves all modal statements in the above modal arrow language, and hence provide a lower level of visual analysis in linear algebra similar to what we have found earlier for topology, or geometry.

Next, logics for valid reasoning also transfer immediately. Here is a display of the basic system of arrow logic:

$$(\varphi \vee \psi) \oplus \xi \leftrightarrow (\varphi \vee \phi) \oplus \xi \quad (13)$$

$$\varphi \oplus (\psi \vee \xi) \leftrightarrow (\varphi \oplus \phi) \vee (\varphi \oplus \xi) \quad (14)$$

$$\neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \vee \neg\psi \quad (15)$$

$$\varphi \wedge (\psi \oplus \xi) \rightarrow \psi \oplus (\xi \wedge (\neg\psi \oplus \varphi)) \quad (16)$$

These principles either represent or imply obvious vector laws. Here are some consequences of (15), (16):

$$\begin{aligned} -(\neg A) &\leftrightarrow \neg(-A) \\ -(A + B) &\leftrightarrow -B + -A \\ A + \neg(-A + \neg B) &\rightarrow B \end{aligned}$$

The latter ‘triangle inequality’ is the earlier rule of Modus Ponens in disguise. On top of this, special arrow logics have been axiomatized with a number of additional frame conditions. In particular, the vector space interpretation makes composition *commutative* and *associative*, which leads to further axioms:

$$\begin{aligned} A \oplus B &\leftrightarrow B \oplus A && \text{commutativity} \\ A \oplus (B \oplus C) &\leftrightarrow (A \oplus B) \oplus C && \text{associativity} \end{aligned}$$

These additional principles make the calculus simpler in some ways than basic arrow logic. The key fact about composition is now the vector law

$$a = b + c \quad \text{iff} \quad c = a - b$$

which derives the triangle inequality. And there are also expressive gains. E.g., the modal language becomes automatically ‘versatile’ in our earlier sense.

Again the soundness of the given arrow logic for vector algebra is clear, and we can freely derive old and new laws of vector algebra. But the central open question about arrow logic and mathematical morphology is again a converse:

What is the complete axiomatization of arrow logic over the standard vector spaces \mathbb{R}^n ?

In particular, are there differences of dimensionality that show up in different arrow principles across these spaces?

Continuing with earlier topics, *extending* the basic modal language of arrows also makes sense. E.g., in general arrow logic there may be many identity arrows, while in vector space there is only one identity element 0. To express this uniqueness, we need to move to some form of modal *difference logic* (cf. Section 2.2). Also, in mathematical morphology, one finds a device for stating laws that are not valid in general, but only when we interpret some variables as standing for single vectors. An example is:

$$\begin{aligned} (X)_t - Y &= (X - Y)_t && \text{(MM-form)} \\ B \rightarrow (A + t) &\Leftrightarrow (B \rightarrow A) + t && \text{(LL-form)} \end{aligned}$$

From right to left, this is **LL** derivable as the general law $(S \rightarrow X) \bullet Y \Rightarrow S \rightarrow (X \bullet Y)$. The converse of this is not **LL** derivable, but it only works when Y is a

singleton $\{t\}$. In the latter case, we have the special principle $S \Leftrightarrow (S + \{t\}) - \{t\}$, which we have to ‘inject’ into an otherwise fine **LL** derivation to get the desired result. This trick is exactly the same as using so-called *nominals* in extended modal logics, cf. [ARE 00], which are special proposition letters denoting just a single point. Other natural language extensions include an infinitary version of the addition modality \oplus , allowing us to close sets to *linear subspaces*.

Thus, the two fields are related, not just in their general structure, but also in their modus operandi, including tricks for boosting expressiveness. Of course, one hopes that the *algorithmic* content of arrow logics also makes sense under this connection, including its philosophy of ‘taming complexity’. This brings us to our final topic:

5.2.3. A worry about complexity

Issues of decidability and complexity have been largely ignored in this paper. But one part of the ‘modal program’ is the balance between moderate expressive power and low complexity for various tasks: model checking, model comparison, and logical inference. In particular, arrow logics were originally designed to make the spectacular jump from undecidability in standard relational algebra to decidability. What happens to arrow logics in mathematical morphology? Even though the logic of the standard models appears to be effectively axiomatizable, i.e., recursively enumerable, *undecidability* is lurking! One bad omen is the validity of associativity, a danger sign in the arrow philosophy (cf. van Benthem 1996). But more precisely, [AIE 02a] shows how to effectively encode an undecidable *tiling problem* [HAR 83] into the complete arrow logic of the two-dimensional Euclidean plane, showing that we may have gone overboard in our desire to express the truth about vectors. Thus, the Balance remains a continuing concern.

6. Concluding Remarks

Our walk through space has shown modal structures wherever one looks. There are natural fine-structured modal versions of topology, affine and metric geometry, and linear algebra. These can be studied by general modal techniques—though much of the interest comes from paying attention to special spatial features. The benefits of this may be uniformity and greater sensitivity to expressive and computational fine-structure in theories of space. But it will also be clear that, in these new waters, we have just charted little islands of knowledge in an ocean of ignorance. Even without stating a huge list of open problems of expressiveness, complexity and complete axiomatization, reading this paper will make it clear that there is any amount of logical work to be done!

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7. References

- [AIE 01] AIELLO M., VAN BENTHEM J., BEZHANISHVILI G., “Reasoning about Space: the Modal Way”, report num. PP-2001-18, 2001, ILLC, University of Amsterdam, Submitted to the *Journal of Logic and Computation*.
- [AIE 02a] AIELLO M., “Spatial Reasoning: Theory and Practice”, PhD thesis, University of Amsterdam, 2002, DS-2002-02.
- [AIE 02b] AIELLO M., “A Spatial Similarity based on Games: Theory and Practice”, *Journal of the Interest Group in Pure and Applied Logic*, vol. 10, num. 1, 2002, p. 1–22, Oxford University Press.
- [AIE 02c] AIELLO M., VAN BENTHEM J., “Logical Patterns in Space”, BARKER-PLUMMER D., BEAVER D., VAN BENTHEM J., SCOTTO DI LUZIO P., Eds., *Words, Proofs, and Diagrams*, CSLI, Stanford, 2002, To appear.
- [ALL 83] ALLEN J., “Maintaining knowledge about temporal intervals”, *Communications of the ACM*, vol. 26, 1983, p. 832–843.
- [ALL 85] ALLEN J., HAYES P., “A common sense theory of time”, JOSHI A., Ed., *IJCAI85*, vol. 1, International Joint Conference on Artificial Intelligence, Morgan Kaufmann, 1985, p. 528–531.
- [ARE 00] ARECES C., “Logic Engineering: The Case of Description and Hybrid Logics”, PhD thesis, University of Amsterdam, 2000.
- [ASH 95] ASHER N., VIEU L., “Toward a Geometry of Common Sense: a semantics and a complete axiomatization of mereotopology”, *IJCAI95*, International Joint Conference on Artificial Intelligence, 1995, p. 846–852.
- [BAL 97] BALBIANI P., FARIÑAS DEL CERRO L., TINCHEV T., VAKARELOV D., “Modal logics for incidence geometries”, *Journal of Logic and Computation*, vol. 7, 1997, p. 59–78.
- [BAL 98] BALBIANI P., “The modal multilogic of geometry”, *Journal of Applied Non-Classical Logics*, vol. 8, 1998, p. 259–281.
- [BEN] VAN BENTHEM J., “Logical structures in mathematical morphology”, Available at <http://www.science.uva.nl/~johan/MM-LL.ps>.
- [BEN 83a] VAN BENTHEM J., “Correspondence Theory”, GABBAY D., GUENTHNER F., Eds., *Handbook of Philosophical Logic*, vol. II, p. 167–247, Reidel Publishing Company, 1983.
- [BEN 83b] VAN BENTHEM J., *The Logic of Time*, vol. 156 of *Synthese Library*, Reidel, Dordrecht, 1983, [Revised and expanded, Kluwer, 1991].
- [BEN 83c] VAN BENTHEM J., DOETS K., “Higher-Order Logic”, GABBAY D., GUENTHNER F., Eds., *Handbook of Philosophical Logic*, vol. 1, p. 275–329, Reidel Publishing Company, 1983.
- [BEN 91a] VAN BENTHEM J., *Language in Action. Categories, Lambdas and Dynamic Logic*, North-Holland, Amsterdam, 1991.
- [BEN 91b] VAN BENTHEM J., “Logic and the flow of information”, PRAWITZ D., SKYRMS B., WESTERTAL D., Eds., *Proceedings of the 9th International Conference of Logic, Methodology and Philosophy of Science*, Elsevier, 1991, p. 693–724.

- [BEN 95] BENNETT B., “Modal Logics for Qualitative Spatial Reasoning”, *Bulletin of the IGPL*, vol. 3, 1995, p. 1–22.
- [BEN 96] VAN BENTHEM J., *Exploring Logical Dynamics*, vol. 156, CSLI Publications, Stanford | Cambridge University Press, 1996.
- [BEN 02] VAN BENTHEM J., BEZHANISHVILI G., GEHRKE M., “Euclidean Hierarchy in Modal Logic”, report num. PP-2002-8888888888888888, 2002, Univ. of Amsterdam.
- [BLA 01] BLACKBURN P., DE RIJKE M., VENEMA Y., *Modal Logic*, Cambridge University Press, 2001.
- [BLO 00] BLOCH I., “Using Mathematical Morphology Operators as Modal Operators for Spatial Reasoning”, *ECAI 2000, Workshop on Spatio-Temporal Reasoning*, 2000, p. 73–79.
- [BLU 61] BLUMENTHAL L., *A Modern View of Geometry*, Dover, 1961.
- [BUR 84] BURGESS J., “Basic Tense Logic”, GABBAY D., GUENTHNER F., Eds., *Handbook of Philosophical Logic*, vol. II, chapter 2, p. 89–133, Reidel, 1984.
- [CAS 99] CASATI R., VARZI A., *Parts and Places*, MIT Press, 1999.
- [CLA 81] CLARKE B., “A calculus of individuals based on ‘connection’”, *Notre Dame Journal of Formal Logic*, vol. 23, num. 3, 1981, p. 204–218.
- [DAB 96] DABROWSKI A., MOSS A., PARIKH R., “Topological reasoning and the logic of knowledge”, *Annals of Pure and Applied Logic*, vol. 78, 1996, p. 73–110.
- [DOE 87] DOETS K., “Completeness and Definability. Applications of the Ehrenfeucht Game in Second-Order and Intensional Logic”, PhD thesis, ILLC, University of Amsterdam, 1987.
- [GAB 94] GABBAY D., HODKINSON I., REYNOLDS M., *Temporal Logic: Mathematical Foundations and Computational Aspects*, vol. 1, Oxford University Press, 1994.
- [GÄR 00] GÄRDENFORS P., *Conceptual Spaces*, MIT Press, 2000.
- [GEO 93] GEORGATOS K., “Modal Logics for Topological Spaces”, PhD thesis, City University of New York, 1993.
- [GIR 87] GIRARD J.-Y., “Linear logic”, *Theoretical Computer Science*, vol. 50, 1987, p. 1–102.
- [GOL 87] GOLDBLATT R., *Orthogonality and Spacetime Geometry*, Springer-Verlag, 1987.
- [GOR 92] GORANKO V., PASSY S., “Using the universal modality: gains and questions”, *Journal of Logic and Computation*, vol. 2, 1992, p. 5–30.
- [HAM 95] HAMMER E., *Logic and Visual Information*, CSLI and FoLLi, Stanford, 1995.
- [HAR 83] HAREL D., “Recurring Dominoes: Making the highly undecidable highly understandable”, *Conference on Foundations of Computer Science*, vol. LNCS 158, Springer, 1983, p. 177–194.
- [HEI 94] HEIJMANS H. J. A. M., *Morphological Image Operators*, Academic Press, Boston, 1994.
- [HEL 23] HELLY E., “Über Mengen konvexer Körper mit gemeinschaftlichen Punkten”, *Jahresber. Deutsch. Math.- Verein*, vol. 32, 1923, p. 175–176.
- [KAM 68] KAMP J., “Tense logic and the theory of linear order”, PhD thesis, University of California, Los Angeles, 1968.

- [KER 01] KERDILES G., “Saying It with Pictures: a logical landscape of conceptual graphs”, PhD thesis, University of Amsterdam, 2001.
- [KUR 95] KURTONINA N., “Frames and Labels. A Modal Analysis of Categorical Inference”, PhD thesis, ILLC, Amsterdam, 1995.
- [KUR 97] KURTONINA N., DE RIJKE M., “Bisimulations for Temporal Logic”, *Journal of Logic, Language and Information*, vol. 6, 1997, p. 403–425.
- [LES 83] LESNIEWSKI S., “The foundation of mathematics”, *Topoi*, vol. 2, 1983, p. 7–52, Translation of the paper of 1931.
- [LEW 73] LEWIS D., *Counterfactuals*, Harvard University Press, 1973.
- [MAR 97] MARX M., VENEMA Y., *Multi Dimensional Modal Logic*, Kluwer, 1997.
- [MAT 67] MATHERON G., *Éléments pur une théorie des milieux poreux*, Masson, 1967.
- [MCK 44] MCKINSEY J., TARSKI A., “The Algebra of Topology”, *Annals of Mathematics*, vol. 45, 1944, p. 141–191.
- [MIN 98] MINTS G., “A Completeness Proof for Propositional S4 in Cantor space”, E. ORLOWSKA, Ed., *Logic at work : Essays dedicated to the memory of Helena Rasiowa*, Physica-Verlag, Heidelberg, 1998.
- [NUT 83] NUTE D., “Conditional Logic”, GABBAY D., GUENTHNER F., Eds., *Handbook of Philosophical Logic*, vol. IV, p. 387–439, Reidel Publishing Company, 1983.
- [PLA 95] VON PLATO J., “The Axioms of Constructive Geometry”, *Annals of Pure and Applied Logic*, vol. 76, num. 2, 1995, p. 169–200.
- [PRE 85] PREPARATA F., SHAMOS M., *Computational Geometry: An Introduction*, Springer-Verlag, 1985.
- [RAN 92] RANDELL D., CUI Z., COHN A. G., “A Spatial Logic Based on Regions and Connection”, *Proc. of Int. Conf. on Principles of Knowledge Representation and Reasoning (KR'92)*, San Mateo, 1992, p. 165–176.
- [RAN 01] RANDELL D., WITKOWSKI M., SHANAHAN M., “From Images to Bodies: Modelling and Exploiting Occlusion and Motion Parallax”, *Proc. of Int. Joint Conference on Artificial Intelligence (IJCAI-01)*, 2001.
- [RAS 63] RASIOWA H., SIKORSKI R., *The Mathematics of Metamathematics*, Państwowe Wydawnictwo naukowe, 1963.
- [RIJ 93] DE RIJKE M., “Extended Modal Logic”, PhD thesis, ILLC, University of Amsterdam, 1993.
- [SEG 70] SEGERBERG K., “Modal Logics with linear alternative relations”, *Theoria*, vol. 36, 1970, p. 301–322.
- [SER 82] SERRA J., *Image Analysis and Mathematical Morphology*, Academic Press, 1982.
- [SHE 99] SHEHTMAN V., ““Everywhere” and “Here””, *Journal of Applied Non-Classical Logics*, vol. 9, num. 2-3, 1999, p. 369–379.
- [STE 00] STEBLETSOVA V., “Algebras, relations and Geometries”, PhD thesis, University of Utrecht, 2000.
- [SZC 65] SZCZERBA L., TARSKI A., “Metamathematical properties of some affine geometries”, BAR-HILLEL Y., Ed., *Int. Congress for Logic, Methodology, and Philosophy of Science*, North-Holland, 1965, p. 166–178.

- [TAR 38] TARSKI A., “Der Aussagenkalkül und die Topologie”, *Fund. Math.*, vol. 31, 1938, p. 103–134.
- [TAR 59] TARSKI A., “What is Elementary Geometry?”, L. HENKIN AND P. SUPPES AND A. TARSKI, Ed., *The Axiomatic Method, with Special Reference to Geometry and Physics*, p. 16–29, North-Holland, 1959.
- [TRO 92] TROELSTRA A. S., *Lectures on Linear Logic*, CSLI, 1992.
- [VEL 85] VELTMAN F., “Logics for conditionals”, PhD thesis, University of Amsterdam, 1985.
- [VEN 92] VENEMA Y., “Many-Dimensional Modal Logic”, PhD thesis, ILLC, University of Amsterdam, 1992.
- [VEN 96] VENEMA Y., “A crash course in Arrow Logic”, MARX M., MASUCH M., PÓLOS L., Eds., *Arrow Logic and Multimodal Logic*, CSLI, 1996.
- [VEN 99] VENEMA Y., “Points, Lines and Diamonds: a Two-Sorted Modal Logic for Projective Planes”, *Journal of Logic and Computation*, vol. 9, num. 5, 1999, p. 601–621.
- [WHI 29] WHITEHEAD A., *Process and Reality*, The MacMillan Company, New York, 1929.