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Modal Frame Correspondences and Fixed-Points

Dedicated to the memory of Willem Johannes Blok

Abstract. Taking Löb's Axiom in modal provability logic as a running thread, we discuss some general methods for extending modal frame correspondences, mainly by adding fixed-point operators to modal languages as well as their correspondence languages. Our suggestions are backed up by some new results — while we also refer to relevant work by earlier authors. But our main aim is advertizing the perspective, showing how modal languages with fixed-point operators are a natural medium to work with.

Keywords: Löb's Axiom, fixed-point, frame correspondence, modal μ-calculus.

1. Introduction: easy and hard correspondences

The topic of this paper goes back to the mid 1970s, when a young Amsterdam logic circle including Wim Blok, Dick de Jongh and the present author, with visitors such as Craig Smorynski, was picking up an interest in modal logic. One special interest in those days has remained important since, viz.

$$\Box(\Box p \to p) \to \Box p \qquad \text{(Löb's Axiom)}$$

in the provability logic of arithmetic. This principle was discovered by Martin Löb, then one of our senior professors. At the time, I had just started working on modal correspondence theory for analyzing the relational frame content of modal axioms. This method works quite neatly for the usual modal axioms such as

$$\Box p \to \Box\Box p \qquad \text{(K4 Axiom)}$$

Let us call a modal formula ϕ true at a point s in a frame $F = (W, R)$ if it is true at s under all atomic valuations V on F. Here is perhaps the most famous correspondence observation:

Fact 1.1. $F, s \models \Box p \to \Box\Box p$ iff F's accessibility relation R is transitive at the point s: i.e., $F, s \models \forall xyz((Rxy \land Ryz) \to Rxz)$.

Proof. If the relation is transitive, $\Box p \to \Box\Box p$ clearly holds under every valuation. Conversely, let $F, s \models \Box p \to \Box\Box p$. In particular, the K4-axiom

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will hold if we take $V(p)$ to be $\{y|\ Rsy\}$. But then, the antecedent $\Box p$ holds at s, and hence so does the consequent $\Box\Box p$. And the latter states the transitivity, by the definition of $V(p)$. ■

Now Löb's Axiom was a challenge, as it does not fit this simple pattern of first-order analysis. One day in 1973, I found a semantic argument for its correct correspondence:

FACT 1.2. Löb's Axiom is true at point s in a frame $F = (W, R)$ iff

(a) F is upward R-well-founded starting from s, and also

(b) F is transitive at s.

PROOF. First, Löb's Axiom implies transitivity. Let Rsx and Rxy, but not Rsy. Setting $V(p) = W - \{x, y\}$ makes Löb's Axiom false at s. Next, let (b) hold. If (a) fails, there is an ascending sequence $s = s_0Rs_1Rs_2\dots$ and setting $V(p) = W - \{s_0, s_1, s_2, \dots\}$ refutes Löb's Axiom at s. Conversely, suppose that Löb's Axiom fails at the world s. In case of a failure of transitivity, there is nothing to be proved. Otherwise, there is an infinite upward sequence of $\neg p$-worlds. This arises by taking any successor of s where p fails, and repeatedly applying the truth of $\Box(\Box p \to p)$ — using the transitivity of the frame at s. ■

Here the transitivity clause (b) was surprising, as the modal $K4$-axiom had always been postulated separately in provability logic. The next day, Dick de Jongh came up with a beautiful purely modal derivation of the transitivity axiom from Löb's Axiom. It revolved around one well-chosen substitution instance

$$\Box(\Box(\Box p \wedge p) \to (\Box p \wedge p)) \to \Box(\Box p \wedge p)$$

REMARK 1. *Consequence via substitution*
In the preceding semantic frame argument deriving clause (b), there is a matching substitution of a first-order definable predicate for the proposition letter p:

$$V(p) := \{y|\ Rsy \wedge \forall z(Ryz \to Rsz)\}$$

With this choice of a predicate p, $\Box(\Box p \to p)$ holds at s, and hence so does $\Box p$ by the validity of the Löb Axiom - and transitivity then follows by the definition of $V(p)$. This theme of modal deduction via suitable set-based substitutions is pursued more systematically in [2].

Later, Wim Blok got into the game, and found the much more complex derivation of $\Box p \to \Box\Box p$ from Grzegorczyk's Axiom, the counterpart of Löb's Axiom on reflexive frames, using algebraic methods. (Cf. [14].) This was to have been one of many illustrations in a planned joint book on modal logic and universal algebra, commissioned by Anne Troelstra for “Studies in Logic” as a merge of our dissertations. The book never happened, though chapter drafts are still lying around. Our friendship remained, however, from those days until Wim's passing away.

The present paper collects a few observations about the behaviour of Löb's Axiom, as a stepping stone for exploring some generalizations of modal frame correspondence. My main concern is whether the usual correspondence arguments can deliver more than they do in their traditional formulation. I think they do.

2. Modal correspondence: from first-order to fixed-points

Let us look first at the general reasoning behind the above $K4$-example.

2.1. Frame correspondence by first-order substitutions

Here is a result from [21], discovered independently in [4]:

THEOREM 2.1. *There is an algorithm computing first-order frame correspondents for modal formulas $\alpha \to \beta$ with an antecedent α constructed from atoms prefixed by universal modalities, conjunction, disjunction, and existential modalities, and the consequent β any syntactically positive formula.*

The translation algorithm for obtaining first-order frame properties from this kind of axiom works as follows:

1. *Translate* the modal axiom into its standard first-order form, prefixed with monadic quantifiers for proposition letters: $\forall x \forall \boldsymbol{P}\ ST(\phi)(\boldsymbol{P}, x)$
2. Pull out all existential modalities occurring in the antecedent, and turn them into bounded universal quantifiers in the prefix,
3. Compute a first-order *minimal valuation* for the proposition letters making the remaining portion of the antecedent true,
4. *Substitute* this definable valuation for the proposition letters occurring in the body of the consequent — and if convenient,
5. Perform some simplifications modulo logical equivalence.

For details of this 'substitution algorithm' and a proof of its semantic correctness, cf. [12]. Here we only provide a sample calculation to demonstrate the method. Consider our initial example:

EXAMPLE 1. For the modal transitivity formula $\Box p \to \Box\Box p$,

1. yields $\forall x : \forall P : \forall xy(Rxy \to Py) \to \forall z(Rxz \to \forall u(Rzu \to Pu))$,
2. is vacuous, while

3. yields the minimal valuation $Ps := Rxs$ | and then
4. substitution yields the initial form $\forall x : \forall xy(Rxy \rightarrow Rxy) \rightarrow \forall z(Rxz \rightarrow \forall u(Rzu \rightarrow Rxu))$.
5. The latter then simpli¯es to the usual form of transitivity $\forall x : \forall z(Rxz \rightarrow \forall u(Rzu \rightarrow Rxu))$.

Concrete modal principles not covered by the substitution method are LÄob's Axiom | and also the following formula, whose antecedent typically has the wrong form:

$$\Box\Diamond p \rightarrow \Diamond\Box p \qquad \text{(McKinsey Axiom)}$$

The McKinsey Axiom is not ¯rst-order de¯nable ([4], [5]).

2.2. An excursion on scattering

The substitution method is quite strong. In particular, the above procedure also works if all modalities are entirely *independent*, as in the following variant of the K4-axiom:

FACT 2.2. $\Box_1 p \rightarrow \Box_2\Box_3 p$ also has a ¯rst-order frame correspondent, computed in exactly the same fashion, viz. $\forall x : \forall z(R_2xz \rightarrow \forall u(R_3zu \rightarrow R_1xu))$.

Here is the relevant general notion.

DEFINITION 1. The *scattered version* of a modal formula ϕ arises by marking each modality in ϕ uniquely with an index for its own accessibility relation.

The Sahlqvist Theorem applies to the scattered version of any implication of the above sort. The reason is that its conditions make statements about individual occurrences: they do not require pairwise co-ordination of occurrences. This sort of condition is frequent in logic, and hence many results have more general scattered versions. Scattering is of interest for several reasons. It suggests *most general versions* of modal results | and the interplay of many di®erent modalities in a single formula ¯ts with the current trend toward *combining logics*. E.g., in provability logic, di®erent boxes could stand for the provability predicates of di®erent arithmetical theories | not just Peano Arithmetic. Even so, scattering does not always apply:

THEOREM 2.3. *There are first-order frame-definable modal formulas whose scattered versions are not first-order frame-definable.*

PROOF. Consider the ¯rst-order de¯nable modal formula which conjoins the K4 transitivity axiom with the McKinsey Axiom (cf. [5]):

$$(\Box p \rightarrow \Box\Box p) \wedge (\Box\Diamond p \rightarrow \Diamond\Box p)$$

Even its partly scattered version $(\Box_1 p \to \Box_1\Box_1 p) \wedge (\Box_2\Diamond_2 p \to \Diamond_2\Box_2 p)$ is not ¯rst-order de¯nable. For, in any frame, taking the universal relation for R_1 will verify the left conjunct, and so, substituting these, the purported total ¯rst-order equivalent would become a ¯rst-order equivalent for the McKinsey axiom: *quod non*. ■

REMARK 2. *Scattering proposition letters.*
One can also make each occurrence of a *proposition letter* unique in modal formulas. But this sort of scattering makes any modal axiom ¯rst-order de¯nable! First, propositionally scattered formulas are either upward or downward monotone in each proposition letter p, depending on the polarity of p's single occurrence. Now [5] shows that modal formulas $\phi(p)$ which are upward (downward) monotone in p are frame-equivalent to $\phi(\bot)$ ($\phi(\top)$). So, propositionally scattered formulas are frame-equivalent to closed ones, and the latter are all ¯rst-order de¯nable.

2.3. Frame correspondence and fixed-point logic

Löb's Axiom is beyond the syntactic range of the Sahlqvist Theorem, as its antecedent has a modal box over an implication. But still, its frame-equivalent of transitivity plus well-foundedness, though not ¯rst-order, is de¯nable in a natural extension | viz. $LFP(FO)$: *first-order logic with fixed-point operators* ([15]).

FACT 2.4. The $well-founded$ part of a binary relation R is the smallest ¯xed-point of the monotone set operator $\Box(X) = \{y|\forall z(Ryz \to z \in X)\}$.

The simple proof is found, e.g., in [1]. The well-founded part can be written in the language of $LFP(FO)$ as the corresponding smallest-¯xed-point formula $\mu P, x. \forall y(Rxy \to Py)$.

How can we ¯nd modal frame equivalents of this extended $LFP(FO)$-de¯nable form as systematically as ¯rst-order frame conditions? The following subsection presents some relevant results from [8] | while the idea of ¯xed-point -based correspondences has also been investigated by di®erent methods in [20], [17]. For a start, Löb's Axiom suggests a general principle, as the *minimal valuation* step in the substitution algorithm still works. Consider the antecedent $\Box(\Box p \to p)$. If this modal formula holds anywhere in a model M, x, then there must be a smallest predicate P for p making it true at M, x | because of the following set-theoretic property guaranteeing a minimal verifying predicate:

FACT 2.5. If $\Box(\Box p_i \to p_i)$ holds at a world x for all $i \in I$, then $\Box(\Box P \to P)$ holds at x for $P = \bigcap_{i\in I}[[p_i]]$.

This fact is easy to check. Here is the more general notion behind this particular observation.

DEFINITION 2. A ¯rst-order formula $\phi(P, \boldsymbol{Q})$ has the *intersection property* if, in every model M, whenever $\phi(P, \boldsymbol{Q})$ holds for all predicates in some family $\{P_i | i \in I\}$, it also holds for the intersection, that is: $M, \bigcap_{i \in I} P_i \models \phi(P, \boldsymbol{Q})$.

Now, the Löb antecedent displays a typical syntactical format which ensures that the intersection property must hold. We can specify this more generally as follows.

DEFINITION 3. A ¯rst-order formula is a *PIA condition* | short for: 'positive antecedent implies atom'- if it has the following syntactic form:

$\forall x(\phi(P, \boldsymbol{Q}, x) \rightarrow Px)$ with P occurring only positively in $\phi(P, \boldsymbol{Q}, x)$.

EXAMPLE 2. *Löb's Axiom*
Translating the antecedent $\Box(\Box p \rightarrow p)$ yields the ¯rst-order PIA form $\forall y((Rxy \wedge \forall z(Ryz \rightarrow Pz)) \rightarrow Py)$.

EXAMPLE 3. *Horn clauses.*
A simpler case of the PIA format is the universal Horn clause de¯ning modal accessibility via the transitive closure of a relation R: $Px \wedge \forall y \forall z((Py \wedge Ryz) \rightarrow Pz))$. The minimal predicate P satisfying this consists of all points R-reachable from x.

It is easy to see that this special syntactic format implies the preceding semantic property:

FACT 2.6. PIA-conditions imply the Intersection Property.

By way of background, here is a model-theoretic preservation result:

THEOREM 2.7. *The following two assertions are equivalent for all first-order formulas* $\phi(P, \boldsymbol{Q})$*:*

1. $\phi(P, \boldsymbol{Q})$ *has the Intersection Property w.r.t. predicate* P
2. $\phi(P, \boldsymbol{Q})$ *is definable by a conjunction of* PIA *formulas.*

For our purposes, we rather need to know what minimal predicates de¯ned using the Intersection Property look like. Here, standard ¯xed-point logic provides an answer:

FACT 2.8. The minimal predicates for PIA-conditions are all de¯nable in the language $LFP(FO)$.

Example 3 was an illustration, as the transitive closure of an accessibility relation is typically de¯nable in $LFP(FO)$. Here is another:

EXAMPLE 4. *Computing the minimal valuation for Löb's Axiom.*
Analyzing $\Box(\Box p \to p)$ a bit more closely, the minimal predicate satisfying the antecedent of Löb's Axiom at a world x describes this set of worlds:

$$\{y|\forall z(Ryz \to Rxz) \text{ \& no infinite sequence of } R\text{-successors starts from } y\}$$

Then, if we substitute this second-order description into the Löb consequent $\Box p$, precisely the usual, earlier-mentioned conjunctive frame condition will result automatically.

Now, plugging these conditions into the above substitution algorithm yields an extension of the earlier Sahlqvist Theorem with respect to a broader class of frame correspondents:

THEOREM 2.9. *Modal axioms with modal PIA antecedents and syntactically positive consequents all have their corresponding frame conditions definable in $LFP(FO)$.*

2.4. Further illustrations, and limits

This extended correspondence method covers much more than the two examples so far. Here are a few more examples of PIA-conditions, in variants of Löb's Axiom.

EXAMPLE 5. *Two simple Löb variants.*

1. With the formula $\Box(\Diamond p \to p) \to \Box p$, the relevant smallest fixed-point for p in the antecedent is defined by $\mu P, y.\, Rxy \wedge \exists z(Ryz \wedge Pz)$, with x the current world. This evaluates to the Falsum $\bot$, and indeed the formula $\Box(\Diamond p \to p) \to \Box p$ is frame-equivalent to $\Box\bot$, as may also be checked directly.
2. The well-known frame-incomplete `Henkin variant' of Löb's Axiom reads as follows: $\Box(\Box p \leftrightarrow p) \to \Box p$. This may be rewritten equivalently as $(\Box(\Box p \to p) \wedge \Box(p \to \Box p)) \to \Box p$. Here, the antecedent is a conjunction of PIA-forms, and unpacking these as above yields the minimal fixed-point formula $\mu P, y.\, (Rxy \wedge \forall z(Ryz \to Pz)) \vee \exists z(Rxz \wedge Pz \wedge Rzy))$.

   But also, *scattering* makes sense again to obtain greater generality:

FACT 2.10. The modal formula $\Box_1(\Box_2 p \to p) \to \Box_3 p$ is equivalent on arbitrary frames $F = (W, R_1, R_2, R_3)$ to the conjunction of the following two relational conditions:

(a) $R_3; (R_2)^* \subseteq R_1$ (with $(R_2)^*$ the reflexive-transitive closure of R_2)

(b) upward well-foundedness in the following sense: no world x starts an infinite upward sequence of worlds $xR_3y_1R_2y_2R_2y_3\ldots$

PROOF. Scattered LÄob implies the generalized transitivity (a) much as it implied transitivity before. Next, assuming the truth of (a), it is easy to see that any failure of scattered LÄob produces an in¯nite upward y-sequence as forbidden in (b), while conversely, any valuation making p false only on such an in¯nite y-sequence will falsify the scattered LÄob Axiom at the world x. ■

REMARK 3. Fact 2.10 arose out of an email exchange with Chris Steinsvold (CUNY, New York), who had analyzed the partially scattered axiom $\Box_1(\Box_2 p \rightarrow p) \rightarrow \Box_1 p$. The general correspondence was also found independently by Melvin Fitting.

But we can also look at quite di®erent modal principles in the same way.

FACT 2.11. The modal axiom $(\Diamond p \wedge \Box(p \rightarrow \Box p)) \rightarrow p$ has a PIA antecedent whose minimal valuation yields the $LFP(FO)$-frame-condition that, whenever Rxy holds, x can be reached from y by some ¯nite sequence of successive R-steps.

The complexity of the required substitutions can still vary considerably here, depending on the complexity of reaching the smallest ¯xed-point for the antecedent via the usual bottom-up ordinal approximation procedure. E.g., obtaining the well-founded part of a relation may take any ordinal up to the size of the model. But for Horn clauses with just atomic antecedents, the approximation procedure will stabilize uniformly in any model by stage ω, and the de¯nitions will be simpler.

Even so, there are limits to the present style of analysis. Not every modal axiom yields to the ¯xed-point approach!

FACT 2.12. The tense-logical axiom expressing Dedekind Continuity is not de¯nable by a frame condition in $LFP(FO)$.

PROOF. Dedekind Continuity holds in the real order $(R, <)$ and fails in the rationals $(Q, <)$. But these two relational structures validate the same $LFP(FO)$-sentences, as there is a *potential isomorphism* between them, for which such sentences are invariant. ■

Returning to the modal language, one often views the LÄob antecedent $\Box(\Box p \rightarrow p)$ and the McKinsey antecedent $\Box\Diamond p$ as lying at the same level of complexity, beyond Sahlqvist forms. But in the present generalized analysis of minimizable predicates, the latter seems much more complicated than the former! Indeed, Valentin Goranko and the present author just found a proof for the following assertion:

FACT 2.13. The McKinsey Axiom $\Box\Diamond p \rightarrow \Diamond\Box p$ has no $LFP(FO)$-de¯nable frame correspondent.

The proof uses two observations. First, [4] showed how the McKinsey Axiom is true on some uncountable frame, without being true in any of its countable elementary subframes. But an argument due to Flum in abstract model theory shows that $LFP(FO)$ does satisfy the strong downward Löwenheim-Skolem Theorem.

3. Modal fixed-point languages

A conspicuous trend in modal logic has been the strengthening of modal languages to remove expressive de¯cits of the base with just $\Box$, $\Diamond$. This re°ects a desire for logic design with optimal expressive power, no longer hampered by the peculiarities of weaker languages bequeathed to us by our frugal ancestors. But then, it makes sense to 'restore a balance'. The above frame correspondence language for natural modal axioms involves $LFP(FO)$ which adds ¯xed-point operators to ¯rst-order logic. So let us extend the modal language itself as well, and work with ¯xed-points on both sides.

3.1. The modal μ-calculus

One such extended language ¯ts very well with Section 2. It is the modal μ-calculus | the natural modal fragment of $LFP(FO)$, and a natural extension of propositional dynamic logic. [18] has a quick tour of its syntax, semantics, and axiomatics. This powerful formalism can de¯ne smallest ¯xed-points in the format

$$\mu p.\, \phi(p) \text{ provided that } p \text{ occurs only positively in } \phi$$

This adds general syntactic recursion to the basic modal language, with no assumption on the accessibility order.

Definition 4. *Fixed-point semantics.* In any model M, the formula $\phi(p)$ with only positive occurrences of the proposition letter p de¯nes an inclusion-monotone set transformation

$$F_\phi(X) = \{s \in X | (M, p := X), s \models \phi\}$$

By the Tarski-Knaster Theorem, the operation F_ϕ must have a smallest ¯xed-point. This can be reached bottom-up by ordinal approximation stages

$$\phi^0, \ldots, \phi^\alpha, \phi^{\alpha+1}, \ldots, \phi^\lambda, \ldots$$
$$\text{with } \phi^0 = \varnothing, \phi^{\alpha+1} = F_\phi(\phi^\alpha), \text{ and } \phi^\lambda = \bigcup_{\alpha<\lambda} \phi^\alpha$$

The smallest ¯xed-point formula $\mu p.\, \phi(p)$ denotes the ¯rst 'repetitive' stage where $\phi^\alpha = \phi^{\alpha+1}$. $\lhd$

EXAMPLE 6. *Transitive closure and dynamic logic.*
The μ-calculus can de¯ne a typical transitive closure modality from dynamic logic like 'some ϕ-world is reachable in ¯nitely many R_a-steps', and that even in two versions:

$$\langle a^*\rangle\phi = \mu p.(\phi \vee \langle a\rangle p) \qquad \text{(re°exive-transitive closure)}$$

$$\langle a^*\rangle\phi = \mu p.(\langle a\rangle\phi \vee \langle a\rangle p) \qquad \text{(transitive closure)}$$

EXAMPLE 7. *Well-foundedness again.*
The modal import of Fact 2.4 is this. The smallest ¯xed-point formula $\mu p.\, \Box p$ de¯nes the well-founded part of the accessibility relation for $\Box$ in any modal model.

The μ-calculus also includes greatest ¯xed points $\nu p.\, \phi(p)$, de¯ned as

$$\neg\mu p.\neg\phi(\neg p)$$

Finally, we recall that the μ-calculus is decidable, and that its validities are e®ectively axiomatized by the following two simple proof rules on top of the minimal modal logic K:

$$\mu p.\, \phi(p) \leftrightarrow \phi(\mu p.\, \phi(p)) \qquad \text{(Fixed-Point Axiom)}$$

$$\text{if } \vdash \phi(\alpha) \to \alpha\ \text{, then } \vdash \mu p.\, \phi(p) \to \alpha \qquad \text{(Closure Rule)}$$

3.2. Working with fixed-points in modal logic

This extended formalism is quite workable as a modal language, a feature which is not yet generally appreciated. We will show this practical aspect by means of a few examples.

For convenience, we dualize the above $\langle a^*\rangle\phi$ to a dynamic logic-style modality $\Box^*\phi$ saying that ϕ is true at all worlds reachable in the transitive closure of the accessibility relation R for single $\Box$. The resulting language formalizes earlier correspondence arguments, and it also suggests new variations on modal axioms.

FACT 3.1. $\Box^*(\Box p \to p) \to \Box^* p$ de¯nes just upward well-foundedness of R.

This follows from the earlier correspondence Fact 2.10 for the scattered Löb Axiom. Thus, transitivity needs an additional explicit $K4$-axiom, separating the two aspects of Löb's provability logic explicitly. We will return to this way of stating things later.

Next, here is a formal correspondence argument recast entirely as a formal modal deduction.

EXAMPLE 8. *Scattered Löb Revisited.* The scattered Löb Axiom of Fact 2.10 implied the frame condition that $R_3; (R_2)^* \subseteq R_1$, which corresponds to the modal axiom

$$\Box_1 p \to \Box_3 \Box_2^* p$$

In a dynamic language this is derivable from a scattered Löb Axiom:

(a)	$\Box_1(\Box_2\Box_2^* p \to \Box_2^* p) \to \Box_3\Box_2^* p$	Scattered Löb axiom with $\Box_2^* p$ for p.
(b)	$\Box_2^* p \leftrightarrow (p \wedge \Box_2\Box_2^* p)$	Fixed-point axiom for *.
(c)	$p \to (\Box_2\Box_2^* p \to \Box_2^* p)$	Consequence of (b).
(d)	$\Box_1 p \to \Box_1(\Box_2\Box_2^* p \to \Box_2^* p)$	Consequence of (c).
(e)	$\Box_1 p \to \Box_3\Box_2^* p$	From (a) and (d).

Another illustration of this modal formalization is the original Fact 1.2 itself. It says that Löb's Axiom is equivalent to the K4-axiom plus the μ-calculus axiom $\mu p.\Box p$ for upward well-foundedness. But this can also be shown by pure modal deduction!

THEOREM 3.2. *Löb's Logic is equivalently axiomatized by the two principles:*

(a) $\Box p \to \Box\Box p$ (b) $\mu p.\Box p$

PROOF. From Löb's Logic to (a) was an earlier-mentioned purely modal deduction. Next, (b) is derived as follows. By the fixed-point axiom of the μ-calculus, we have that $\Box\mu p.\Box p \to \mu p.\Box p$. So it suffices to get $\Box\mu p.\Box p$. Now Löb's Axiom implies:

$$\Box(\Box\mu p.\,\Box p \to \mu p.\Box p) \to \Box\mu p.\,\Box p$$

and the antecedent of this is derivable by modal Necessitation from the converse of the μ-calculus fixed-point axiom. Next, assume (a) and (b). We show that, in K4

$$\mu p.\Box p \to (\Box(\Box q \to q) \to \Box q)$$

By the earlier derivation rule for smallest fixed-points, $\mu p.\Box p \to \alpha$ can be proved for any formula α if $\Box\alpha \to \alpha$ can be proved. But we can prove

$$\Box(\Box(\Box q \to q) \to \Box q) \to (\Box(\Box q \to q) \to \Box q)$$

by means of a straightforward derivation in K4. ∎

This re-axiomatization of Löb's Logic only works over a μ-calculus base. For more on the connection between the two logics, see Section 4 below. Continuing with our practical observations, the above version of Löb's Axiom still implies upward well-foundedness, and hence a form of inductive proof over this well-founded order. Thus, there must also be a direct link between Löb's Axiom and the induction axiom of propositional dynamic logic:

$$(\Box\phi \wedge \Box^*(\phi \to \Box\phi)) \to \Box^*\phi \qquad \text{(IND)}$$

FACT 3.3. Löb's Axiom plus the Fixed-Point Axiom $\Box^*\phi \leftrightarrow (\Box\phi \wedge \Box\Box^*\phi)$ (FIX) derive the Induction Axiom of propositional dynamic logic.

PROOF. This can be shown using the above analysis of Löb's Axiom, since the Induction Axiom expresses the greatest fixed-point character of $\Box^*$. An explicit modal derivation is found in the extended version of [9], which points out that earlier published logics of finite trees have a redundant axiom set with full *PDL* plus Löb's Axiom. ■

But one can also recast the link between provability logic and fixed-point logics to the following version:

THEOREM 3.4. *Löb's Logic can be faithfully embedded into the μ-calculus.*

PROOF. The translation doing this works as follows:

1. Replace every $\Box$ in a formula ϕ by its transitive closure version $\Box^*$.
2. For the resulting formula $(\phi)^*$, take the implication $\mu p.\Box p \rightarrow (\phi)^*$.

It is straightforward to check that a plain modal formula ϕ is valid on transitive upward well-founded models iff $\mu p.\Box p \rightarrow (\phi)^*$ is valid on all models. ■

As a consequence, decidability of Löb's Logic follows from that of the μ-calculus. Albert Visser points out that the translation can also be made more compositional:

$$\Box(\phi)^0 = \Box^*(\mu p.\Box p \rightarrow (\phi)^0)$$

Other features may have applications, too, such as the strong interpolation properties of the μ-calculus ([3]). Now, the latter system is more expressive than the usual modal language of provability logic. But this extended setting also raises interesting new issues in the latter area — such as:

QUESTION 1. Can the usual arithmetical interpretation of provability logic be extended to provability logic with a full μ-calculus?

This would require an arithmetical translation respecting the difference between arbitrary well-founded relations R and their transitive closures. The question loses interest, though, in the light of [25]: see Section 4.

3.3. Frame correspondence in extended modal languages

The μ-calculus is just one in a spectrum of extensions of the basic modal language with recursion mechanisms.

Fragments of the μ-calculus. A useful weaker language is propositional dynamic logic (*PDL*) with modalities $\langle\pi\rangle$ for program expressions π constructed out of atomic accessibility relations $a, b, \ldots$ and tests $?\phi$ on arbitrary formulas ϕ, using composition ;, union $\cup$, and iteration * on binary relations. PDL can deal with most of the preceding examples, witness Fact 3.1, which says that a PDL-variant of Löb's Axiom de¯nes $\mu p.\Box p$. Further examples of its expressive power will follow in Section 3.4. Example 6 already showed how PDL is contained in the μ-calculus. Moreover, [18] shows that it is strictly weaker.

FACT 3.5. The ¯xed-point formula $\neg\mu p.\Box p$ (or alternatively, $\nu p.\Diamond p$) is not PDL-de¯nable.

PROOF. This formula de¯nes the set of worlds where some in¯nite R-sequence starts, and this set is not de¯nable in the language of PDL, by a simple semantic argument. ■

Looking top-down, the preceding observation shows that the μ-calculus has natural fragments restricting its powers of recursion. One of these already occurred in Fact 2.11:

DEFINITION 5. The ω-μ-calculus.
The ω-μ-calculus only allows ¯xed-point operators in an existential format, where approximation sequences always stabilize by stage ω:

$$\mu p.\phi(p) \text{ with } \phi \text{ constructed according to the syntax} \quad \text{(i)}$$

$$p \mid p\text{-free formulas } \mid \vee \mid \wedge \mid \text{existential modalities} \quad \text{(ii)}$$

[7] proves a preservation theorem showing the adequacy of this format for the required property of `¯nite distributivity' for the approximation maps. Clearly, PDL is contained in the ω-μ-calculus. But there is a hierarchy:

FACT 3.6. The ω-μ-formula $\mu p.([1]\bot \wedge [2]\top) \vee (\langle 1\rangle p \wedge \langle 2\rangle p)$ is not de¯nable in PDL.

PROOF. (Sketch) This formula expresses that there is a ¯nite binary tree-like submodel starting from the current world, with both R_1- and R_2-daughters at each non-terminal node. Now PDL-formulas only describe reachability along ¯nite traces belonging to some regular language over tests and transitions. This tree property is not like that. ■

Still, PDL is closed under smallest simultaneous ¯xed-points of a yet more special type of recursion, consisting of disjunctions of existential formulas $\langle\pi\rangle p$ where the propositional recursion variables p occurs only in the end position. We omit details here (cf. [10]).

Propositional quantifiers. But there are further relevant extended modal languages. In particular, the μ-calculus is related to the much stronger system $SOML$ of modal logic with second-order quanti¯ers over proposition letters. Cf. [13] for a recent model-theoretic study of $SOML$. Fact 3.1 and Theorem 3.2 suggest the following.

FACT 3.7. The μ-calculus is de¯nable in $SOML$ plus a PDL-style iteration modality $\Box^*$ referring to all worlds accessible from the current one.

PROOF. A smallest ¯xed-point formula $\mu p.\phi(p)$ denotes the intersection of all `pre-¯xed points' of the map $F_\phi(X)$ of De¯nition 4, where $F_\phi(X) \subseteq X$. But the latter set is also de¯ned with one monadic predicate quanti¯cation by the $SOML$-formula $\forall p : \Box^*(\phi(p) \to p) \to p$. ■

The PDL-addition is necessary here, since $SOML$-formulas by themselves have a ¯nite modal depth to which they are insensitive, just like basic modal formulas. Interestingly, the ¯nal formula here is much like that used in [19] to show that PDL with added 'bisimulation quanti¯ers' is expressively equivalent to the μ-calculus.

We conclude with a concrete example that the new formalisms really extend the old.

FACT 3.8. Well-foundedness is not de¯nable in basic modal logic.

PROOF. Suppose that a modal formula ϕ de¯ned well-foundedness. Then it fails at 0 in the frame $(\mathbb{N}, S)$, with S the relation of immediate successor. But then, by the ¯nite depth property of basic modal formulas, ϕ would also fail at 0 in some ¯nite frame $(\{0, .., n\}, S)$, which is well-founded. A similar non-de¯nability argument works for the above formula $p \to \Diamond^* p$, observing that the frames with a partial function R where it holds are just the collections of disjoint ¯nite loops. ■

The same proof shows that well-foundedness is not even de¯nable in $SOML$, as the latter logic still has the ¯nite-depth property.

3.4. Frame correspondences in different fixed-point languages

Compared with the basic theory, languages with modal ¯xed-points support interesting new frame correspondences. Some of these occur inside propositional dynamic logic, others crucially involve the μ-calculus, and eventually, one could look at $SOML$ as well.

EXAMPLE 9. *Cyclic return simplified.*
The formula $p \to \Diamond^* p$ says that every point x is part of some ¯nite R-loop.

EXAMPLE 10. *Term rewriting.*
The formula $\Diamond\Box^* p \to \Box\Diamond^* p$ expresses the Weak Con°uence property that points diverging from a common root have a common successor in the transitive closure of the relation. Basic laws of term rewriting (cf. [11]) then amount to implications between such modally de¯nable graph properties.

These results are subsumed under the following extension of Theorem 2.1. It is by no means the best possible result, but it does show how the original minimal substitution algorithm generalizes.

THEOREM 3.9. *There is an algorithm finding frame-correspondents in LFP(FO) for all modal implications $\alpha \to \beta$ whose consequent β is wholly positive, and whose antecedent α is constructed using*

1. *proposition letters possibly prefixed by universal modalities* $[\pi]$ *in whose PDL-program π all proposition letters occur positively, and over these*
2. $\wedge$, $\vee$, *and existential modalities $\langle\sigma\rangle$ with a test-free PDL-program σ.*

PROOF. (Outline) The main algorithm extracts universal pre¯xes for the $\langle\sigma\rangle$ as in Section 2. Next, the dynamic logic operators $[\pi]$ express modal PIA-conditions, which can be used as a basis for minimization inside $LFP(FO)$. ■

Still, this version seems sub-optimal, as a genuine ¯xed-point version might describe the relevant syntax very di®erently. Cf. [17] for the best available results on correspondence for modal ¯xed-point languages so far.

EXAMPLE 11. *Re-describing modalities.*
From a μ-calculus perspective, a universal modality $[a^*]p$ is a *greatest fixed-point* operator $\nu q.\, p \wedge [a]q$. So, minimizing for p here would compute the formidable-looking iterated ¯xed-point formula $\mu p.\, \nu q.\, p \wedge [a]q$. One then sees that this is equivalent to the set of worlds a^*-reachable from the current world | which can also be described by one μ-type ¯xed-point in $LFP(FO)$.

On the other hand, moving to weaker correspondence languages, one might also drop the universal modalities in Theorem 3.9, and work inside just the ω-μ-calculus or PDL.

QUESTION 2. What is the best possible formulation of the Sahlqvist Theorem in propositional dynamic logic? And in the modal μ-calculus?

[20] also provide a very general correspondence method DLS going back to Ackermann's Lemma in second-order logic. Finally, the $SCAN$ algorithm of [16] also covers both ¯rst- and higher-order cases.

In addition to correspondence issues, there is also modal de¯nability. Many formulas in our examples still satisfy the usual semantic properties of basic modal formulas: they are preserved under generated subframes, disjoint unions, p-morphic images, and anti-preserved under ultra¯lter extensions. The ¯rst three hold for all μ-calculus formulas, by their bisimulation invariance. As for anti-preservation under ultra¯lter extensions, it is easy to see that the usual proof for the basic modal language does not go through, as some sort of in¯nite disjunction splitting would be needed. But we have not been able to ¯nd a counter-example to the property as such. The typical di®erence with basic modal formulas might lie really in the ¯nite evaluation bound of the former, as opposed to even PDL-formulas involving $\Diamond^*$.

These observations suggest various new issues. As an illustration, we state one basic model-theoretic question:

QUESTION 3. Is there a Goldblatt-Thomason Theorem for modal logic with ¯xed-points, saying that the modally de¯nable $LFP(FO)$ frame classes are just those satisfying the stated four semantic preservation properties?

REMARK 4. *Extended languages and expressive completeness.*
Sometimes, a language extension to *tense logic* makes sense to express earlier correspondences compactly. Consider the modal axiom $(\Diamond_a p \wedge \Box_a(p \to \Box_a p)) \to p$ of Fact 2.11, expressing a variant of Cyclic Return. This frame property can also be expressed in propositional dynamic logic with a past tense operator as follows:

$$p \to [a]\langle((PAST\ p)?; a)^*\rangle\langle a\rangle p$$

[23] shows the naturalness of 'versatile' formalisms with converse modalities for the purpose of de¯ning the substitutions of Section 2 inside the modal language. The general point here is that languages with nominals naming speci¯c worlds and backward-looking tense operators de¯ne minimal predicate substitutions, making the modal language expressively complete for its own Sahlqvist correspondences. Cf. also the de¯nability results for frame classes in hybrid languages in [13].

4. An Excursion into Provability Logic

The μ-calculus is perhaps the most natural modal ¯xed-point logic. But there are other, and older, modal ¯xed-point results! This section discusses the linkage between the two grand traditions in modal ¯xed-point logics. Throughout, we will be working in the setting of LÄob's Logic for provability unless otherwise speci¯ed.

4.1. The De Jongh-Sambin fixed-point theorem

A celebrated result in provability logic is the following modal version of the arithmetical Fixed-Point Lemma underlying the proof of GÄodel's Theorem:

THEOREM 4.1. *Consider any modal formula equivalence $\phi(p, \boldsymbol{q})$ in which proposition letters p only occur in the scope of at least one modality, while $\boldsymbol{q}$ is some sequence of other proposition letters. There exists a formula $\psi(\boldsymbol{q})$*

such that $\psi(\boldsymbol{q}) \leftrightarrow \phi(\psi(\boldsymbol{q}), \boldsymbol{q})$ is provable in Löb's Logic, and moreover, any two solutions to this fixed-point equation w.r.t. ϕ are provably equivalent.

For a proof, cf. [22]. This survey paper also gives a simple algorithm for explicitly computing the ¯xed-point $\psi(q)$. Typical outcomes are the following ¯xed points:

EXAMPLE 12. *Solving fixed-point equations in provability logic.*
Here are a few typical cases:

Equation		Solution	
	$p \leftrightarrow \Box p$		$p = \top$
	$p \leftrightarrow \neg\Box p$		$p = \neg\Box\bot$
	$p \leftrightarrow (\Box p \rightarrow q)$		$p = \Box q \rightarrow q$

More complex recursions arise when the body of the modal equation has multiple occurrences of p. Explicit solutions are then obtained by suitably iterating the single-step case.

There are two aspects to Theorem 4.1: (a) existence and uniqueness of the new predicate de¯ned, and (b) explicit de¯nability of that predicate in the modal base language. Here, existence and uniqueness of the predicate p is just a general property of all recursive de¯nitions over well-founded orderings. But we also get the concrete information that this recursive predicate can be de¯ned inside the original modal language, without explicit μ- or ν-operators. Let's compare this with the μ-calculus.

4.2. Provability fixed-points and μ-calculus

We can obviously compare the general approximation procedure of Section 3 and the special-purpose algorithm mentioned just now. For a start, evidently, de¯nitions $\mu p.\phi(p)$ with only positive boxed occurrences of p in ϕ fall under both approaches.

EXAMPLE 13. *The fixed-point for the modal equation $p \leftrightarrow \Box p$.*
$\mu p.\Box p$ de¯ned the well-founded part of the binary relation R. Thus, in well-founded models, it de¯nes the whole universe | which explains Smorynski's solution $\top$ (`true').

But the De Jongh-Sambin Theorem also allows for negative occurrences of p in the de¯ning equation. These fall outside of general ¯xed-point logics.

EXAMPLE 14. *The fixed-point for the modal equation $p \leftrightarrow \neg\Box p$.*
Here, the approximation sequence for the set operator $F_{\neg\Box p}$ can fail to yield a ¯xed point, oscillating all the way. E.g., in the model $(\mathbb{N}, <)$, one gets $\varnothing, \mathbb{N}, \varnothing, \mathbb{N}, \ldots$

Actually, the situation in general ¯xed-point logic is a bit more complex. Formulas with mixed positive and negative occurrences can sometimes be admissible after all.

Example 15. *The mixed-occurrence formula* $p \leftrightarrow (p \vee \neg\Box p)$.
In this case, the approximation sequence will be monotonically non-decreasing, because of the initial disjunct p. So, in any model, there must be a smallest ¯xed-point. With our formula $p \leftrightarrow (p \vee \neg\Box p)$, the sequence stabilizes at stage 2, yielding $\Diamond\top$. There is also a greatest ¯xed-point, which is the whole set de¯ned by $\top$.

This case is beyond Theorem 4.1, as the ¯rst occurrence of p in $p \vee \neg\Box p$ is not boxed. Indeed, there is no unique de¯nability in this extended format, as the smallest and greatest ¯xed-points are di®erent here. In ¯xed-point logic, this example motivates an extension of the monotonic case ([15]).

Definition 6. *Inflationary fixed-points* for arbitrary formulas $\phi(p, \boldsymbol{q})$ without syntactic restrictions on the occurrences of p are computed using an ordinal approximation sequence as above, but now forcing upward cumulation at successor steps:

$$\phi^{\alpha+1} = \phi^{a} \cup \phi(\phi^{a}) \text{ , taking unions again at limit ordinals.}$$

There is no guarantee that a set P where this stabilizes is a ¯xed-point for the modal formula $\phi(p, \boldsymbol{q})$. It is rather a ¯xed-point for the modi¯ed formula $p \vee \phi(p, \boldsymbol{q})$. Instead of in°ationary ¯xed-points, however, one might also use other limit conventions, such as Gupta and Herzberger-style *limsups* and *liminfs* (cf. [24]).

4.3. Combining the two sorts of fixed-point

Comparison may also mean combination. Would *adding* general monotone ¯xed-points to provability logic extend the scope of the De Jongh-Sambin result? The answer is negative.

Fact 4.2. Any p-positive formula $\mu p.\,\phi(p)$ with $\phi(p)$ possibly having unboxed occurrences of p is equivalent to a formula which has all its occurrences of p boxed.

Proof. Without loss of generality, we can take the formula to be of the form $\mu p.(p \wedge A) \vee B$ with only boxed occurrences of p in A, B.

Let ϕ^{α} be the approximation sequence for $\phi = ((p \wedge A) \vee B)$, and let B^{α} be such a sequence executed separately for the formula B. We then have the following collapse:

Lemma 1. $\phi^{\alpha} = B^{\alpha}$ for all ordinals α.

PROOF. This is proved by induction. The zero and limit cases are obvious. Next, we note that

$$\begin{aligned}\phi^{\alpha+1} &= (\phi^{\alpha} \wedge A(\phi^{\alpha})) \vee B(\phi^{\alpha}) \\ &= (B^{\alpha} \wedge A(B^{\alpha})) \vee B(B^{\alpha})\end{aligned}$$

where, by the fact that F_B is monotone: $B^{\alpha} \subseteq B(B^{\alpha})$, and hence $B^{\alpha} \cap A(B^{\alpha}) \subseteq B(B^{\alpha})$

$$\begin{aligned}&= B(B^{\alpha}) \\ &= B^{\alpha+1}\end{aligned}$$

Thus, the same ¯xed-point is computed by the boxed formula $\mu p.B$. ■

The main Fact follows immediately from the Lemma. ■

Albert Visser (p.c.) noted that our arguments so far even establish a sharper result on existence of ¯xed-points:

> any formula $\phi(p)$ in which every occurrence of p is either positive or boxed has a minimal ¯xed-point.

Next, can we ¯t De Jongh-Sambin recursions into general ¯xed-point logic? Recall that well-founded relations have an inductive character: their domains are smallest ¯xed-points de¯ned by $\mu p.\Box p$. On such orders, the whole universe is eventually computed through the monotonically increasing ordinal approximation stages

$$D^0, D^1, .., D^{\alpha}, \ldots$$

of the modal formula $p \leftrightarrow \Box p$. Now we cannot compute similar cumulative stages for the ¯xed-point formula $\phi(p, \boldsymbol{q})$ in Theorem 4.1, as ϕ may have both positive and negative occurrences of the proposition letter p. But we can de¯ne the related monotonic sequence of *inflationary fixed-points*, de¯ned above. As we noted, this in°ationary process need not lead to a ¯xed-point for $\phi(p, \boldsymbol{q})$ *per se*. But this time, we do have monotone growth within the D-hierarchy, as the ϕ's stabilize inside its stages:

FACT 4.3. $\phi^{\alpha+1} \cap D^{\alpha} = \phi^{\alpha} \cap D^{\alpha}$

Thus a general ¯xed-point procedure for solving De Jongh-Sambin equations runs monotonically when restricted to approximation stages for a well-founded universe. This prediction pans out for the above modal examples $\Box p$, $\neg\Box p$, and $\Box p \to q$. We will not prove this here, as we will re-describe the situation now in slightly di®erent terms.

THEOREM 4.4. *De Jongh-Sambin fixed-points can be found by the following simultaneous inflationary inductive definition:*

$$\begin{aligned} r &\leftrightarrow \Box r \\ p &\leftrightarrow \Box r \wedge \phi(p, \boldsymbol{q})\end{aligned}$$

PROOF. We compute the approximation stages for p, r simultaneously:

$$\begin{array}{rcll} (r^{\alpha+1}, p^{\alpha+1}) & = & (\Box r^{\alpha}, \Box r^{\alpha} \wedge \phi(p^{\alpha})) & \text{successors} \\ (r^{\lambda}, p^{\lambda}) & = & (\bigcup_{\alpha<\lambda} r^{\alpha}, \bigcup_{\alpha<\lambda} p^{\alpha}) & \text{limits} \end{array}$$ ■

Here the conjunct $\Box r$ (rather than 'r') for p makes sure that the next stage of p is computed by reference to the new value of r. Now it su±ces to prove the following relation between the approximation stages | written here with some abuse of notation:

LEMMA 2. If $\beta < \alpha$ then $p^{\alpha} \wedge r^{\beta} = p^{\beta}$.

Note that this implies monotonicity: If $\beta < \alpha$ then $p^{\beta} \to p^{\alpha}$.

PROOF. Here, the main induction is best done on α, with an auxiliary one on β. The cases of 0 and limit ordinals are straightforward. For the successor step, we need two auxiliary facts. We state them for arbitrary relations, even though we only use transitive ones here. The ¯rst expresses the invariance of modal formulas for generated submodels, and the second is an immediate consequence of the approximation procedure for r:

(i) $M, P, x \models \phi(p)$ i® $M, P \cap R^{*}[x], x \models \phi(p)$

(ii) Let $R^{*}[x]$ be all points reachable from x by some ¯nite but non-zero number of R-steps. If $x \in r^{\alpha}$, then $R^{*}[x] \subseteq \bigcup_{\beta<\alpha} r^{\beta}$.

Now we compute | again with some bene¯cial abuse of notation:

$$\begin{array}{cl} x \models p^{\alpha+1} \wedge r^{\beta+1} & \text{i®} \\ x \models r^{\alpha+1} \wedge \phi(p^{\alpha}) \wedge r^{\beta+1} & \text{i®} \\ x \models \phi(p^{\alpha}) \wedge r^{\beta+1} & \text{i® (by (i), (ii))} \\ x \models \phi(p^{\alpha} \wedge r^{\beta}) \wedge r^{\beta+1} & \text{i® (by ind. hyp.)} \\ x \models \phi(p^{\beta}) \wedge r^{\beta+1} & \text{i®} \\ x \models p^{\beta+1} & \end{array}$$ ■

4.4. Why the explicit definability?

Our μ-calculus analysis does not explain why provability ¯xed-points are *explicitly definable* in the modal base language. Indeed, the general reason seems unknown. We do know that this phenomenon of explicit de¯nability is not speci¯c to the modal language:

THEOREM 4.5. *Explicit definability for fixed-point equations with all occurrences of p in the scope of some operator holds for all propositional languages with generalized quantifiers Qp over sets of worlds satisfying*

(a) $=$ (i) *above:* $Q(P)$ *is true at* x *iff* $Q(P \cap R_x)$ *is true at* x (Locality)

(b) $Qp \to \Box Qp$ (Heredity)

This covers quanti¯ers Q like the modal "in some successor", the true ¯rst-order "in at most ¯ve successors", or the second-order "in most successors of each successor". [6] has a proof for Theorem 4.5, found in joint work with Dick de Jongh around 1985.

But the general rationale of explicit de¯nability still eludes us. One factor besides appropriate base quanti¯ers Q is transitivity of accessibility. E.g., the Gödel equation $p \leftrightarrow \neg\Box p$ has no explicit modal solution on ¯nite trees with the immediate successor relation. But there may be still deeper model-theoretic reasons for the success of Theorem 4.1 in provability logic in terms of general ¯xed-point logic. Here is a suggestive observation. Smallest and greatest ¯xed points for a *first-order formula* $\phi(P)$ coincide if $\phi(P)$ implies an explicit de¯nition for P. But the converse is true as well, by a straightforward appeal to Beth's Theorem (cf. [22]). Such explicit ¯rst-order de¯nitions for unique ¯rst-order ¯xed-points even arise uniformly by some ¯xed ¯nite approximation stage in every model where they are computed, by the Barwise-Moschovakis Theorem (I owe this reference to Martin Otto).

REMARK 5. *Alternative modal formalisms for solving fixed-point equations.* Visser and d'Agostino have suggested analyzing explicit de¯nability in provability logic with ideas from [19], in particular, uniform interpolation of the μ-calculus, and associated languages with so-called *bisimulation quantifiers*.

4.5. Added in print: provability logic and μ-calculus once more

In a response to an earlier version of this paper (an ILLC Preprint has circulated since early 2005), [25] has made a number of substantial advances. Basically, Visser collects the various observations in the preceding Sections 4.1-3 into one major result:

THEOREM 4.6. *The μ-calculus can be interpreted in Löb's Logic.*

The elegant proof works with categories of interpretations. It also shows that the above interpretation from provability logic in the μ-calculus, and Visser's converse form a *retraction* preserving various model-theoretic properties. Even so, there does not seem to be a *faithful* embedding from the μ-calculus into Löb's Logic. Or, if one exists, it must be somewhat nonstandard, in that the complexities of satis¯ability are di®erent in the two cases: *Pspace*-complete for the latter, and *Exptime*-complete for the former.

5. Higher-order perspectives

Many topics in the preceding sections suggest a further extension into second-order logic, which is the natural habitat of frame truth of modal formulas interpreted as monadic Π^1_1-sentences. For instance, the Sahlqvist Theorem for basic modal logic also works with positive antecedents in any higher-order logic ([7]). But as is well-known, our fixed-point extensions are also fragments of second-order logic. In particular, there might be Beth Theorems for suitable fragments of second-order logic behind the modal fixed-point results discussed in Section 4. [5], [13] study modal logic partly as a way of finding well-behaved fragments of second-order logic. This seems another interesting way to go.

6. Conclusion

This note has shown how various aspects of provability logic, all high-lighted by Löb's Axiom, suggest a much broader background in modal and classical logic, with fixed-point languages as a running thread. Thirty years after our student days, the content of our modal boxes, even in very familiar settings, has not yet been exhausted!

References

[1] Aczel, P., 'An Introduction to Inductive Definitions', in J. Barwise (ed.), *Handbook of Mathematical Logic*, North-Holland, Amsterdam, 1977, pp. 739–782.

[2] d'Agostino, G., J. van Benthem, A. Montanari, and A. Policriti, 'Modal Deduction in Second-Order Logic and Set Theory. Part I', *Logic and Computation* 7 (1997), 251–265.

[3] d'Agostino, G., and M. Hollenberg, 'Logical Questions concerning the μ-Calculus: Interpolation, Lyndon & Lös-Tarski', *Journal of Symbolic Logic* 65 (2000), 310–332.

[4] van Benthem, J., 'Some Correspondence Results in Modal Logic', Report 74–05, Mathematisch Instituut, Universiteit van Amsterdam, 1974.

[5] van Benthem, J., *Modal Logic and Classical Logic*, Bibliopolis, Napoli, 1983.

[6] van Benthem, J., 'Toward a Computational Semantics', in P. Gärdenfors (ed.), *Generalized Quantifiers: Linguistic and Logical Approaches*, Reidel, Dordrecht, 1987, pp. 31–71.

[7] van Benthem, J., 'The Range of Modal Logic', *Journal of Applied Non-Classical Logics* 9, 2/3 (1999), 407–442.

[8] van Benthem, J., 'Minimal Predicates, Fixed-Points, and Definability', Report PP-2004-01, ILLC Amsterdam. Appeared in *Journal of Symbolic Logic* 70, 3 (2005), 696–712.

[9] van Benthem, J., S. van Otterloo, O. Roy, 'Preference Logic, Conditionals, and Solution Concepts in Games', Report PP-2005-28, ILLC Amsterdam, to appear in *Festschrift for Krister Segerberg*, University of Uppsala, 2006.

[10] van Benthem, J., J. van Eijck, B. Kooi 'Logics for Communication and Change', ILLC Preprint DARE 14 8524, University of Amsterdam, in R. van der Meijden, (ed.), *Proceedings of TARK 2005*, National University of Singapore, 2005.

[11] Bezem, J-W., and R. de Vrijer (eds.), ('Terese'), *Term Rewriting Systems*, Cambridge Tracts in Theoretical Computer Science, Vol. 55, Cambridge University Press, 2003.

[12] Blackburn, P., M. de Rijke and Y. Venema, *Modal Logic*, Cambridge University Press, 2001.

[13] ten Cate, B., *Model Theory for Extended Modal Languages*, ILLC Dissertation Series DS-2005-01, University of Amsterdam, 2005.

[14] Blok, W., and J. van Benthem, 'Transitivity Follows from Dummett's Axiom', *Theoria* 44, 2 (1978), 117–118.

[15] Ebbinghaus, H-D., and J. Flum, *Finite Model Theory*, Springer, Berlin, 1995.

[16] Gabbay, D., and H-J. Ohlbach, 'Quantifier Elimination in Second-Order Predicate Logic', *South African Computer Journal* 7 (1992), 35–43.

[17] Goranko, V., and D. Vakarelov, 'Elementary Canonical Formulas I. Extending Sahlqvist's Theorem', Department of Mathematics, Rand Afrikaans University, Johannesburg & Faculty of Mathematics and Computer Science, Kliment Ohridski University, Sofia, 2003.

[18] Harel, D., D. Kozen, and J. Tiuryn, *Dynamic Logic*, The MIT Press, Cambridge (Mass.), 2000.

[19] Hollenberg, M., *Logic and Bisimulation*, Dissertation Series Vol. XXIV, Zeno Institute of Philosophy, University of Utrecht, 1998.

[20] Nonnengart, A., and A. Szalas, 'Fixed-Point Approach to Second-Order Quantifier Elimination with Applications to Modal Correspondence Theory', in E. Orłowska (ed.), *Logic at Work*, Physica-Verlag, Heidelberg, 1999, pp. 89–108.

[21] Sahlqvist, H., 'Completeness and Correspondence in First and Second Order Semantics for Modal Logic', in S. Kanger (ed.), *Proceedings of the Third Scandinavian Logic Symposium*, Amsterdam, North Holland, 1975, pp. 110–143.

[22] Smorynski, C., 'Modal Logic and Self-Reference', in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, Vol. II, Reidel, Dordrecht, 1984, pp. 441–495.

[23] Venema, Y., *Many-Dimensional Modal Logics*, Dissertation, Institute for Logic, Language and Computation, University of Amsterdam, 1991.

[24] Visser, A., 'Semantics and the Liar Paradox', in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, Vol. IV, Reidel, Dordrecht, 1984, pp. 617–706.

[25] Visser, A., 'Löb's Logic Meets the μ-Calculus', Philosophical Institute, University of Utrecht. To appear in *Liber Amicorum for Jan-Willem Klop*, 2005.

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