A Note on **DYNAMIC ARROW LOGIC**

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1   Finding the Computational Core

The current interest in logic and information flow has found its technical expression in various systems of what may be called 'dynamic logic' in some broad sense. But unfortunately, existing dynamic logics based on binary transition relations between computational states have high complexity (cf. Harel 1984). Therefore, it is worthwhile rethinking the choice of a relatively simple dynamic base system forming the 'computational core' that we need, without getting entangled in the complexity engendered by the additional 'mathematics of ordered pairs'. To some extent, this program is realized by various algebraic theories of programs and actions. But the conventional wisdom enshrined in these approaches may be biased: for instance, in its insistence that Boolean negation or complement is the main source of complexity. This may be seen by developing an alternative, namely a modal logic of 'arrows', which takes transitions seriously as dynamic objects in their own right (cf. van Benthem 1991, Venema 1992). The main technical contribution of this Note is the presentation of a system of Arrow Logic with both first-order relational operations and infinitary Kleene iteration, which may be a good candidate for the computational core calculus. In particular, we prove completeness for its minimal version, and establish various connections with propositional dynamic logic and categorial logics.

There is a more general program behind the above proposal. For instance, one can do the same kind of 'arrow analysis' for many other systems in the computational literature, such as the 'dynamic modal logic' in De Rijke's contribution to this volume. Moreover, issues of apparent undecidability and higher-order complexity abound in the semantics of programming. For instance, in Hoare Logic, infinitary control structures create high complexity: is this inevitable, or can the situation be mitigated by redesign? Likewise, in Knowledge Representation, higher-order data structures (such as 'branches' in trees) can generate complexity, witness the field of 'branching temporal logic', which may be avoided by suitable re-analysis in many-sorted first-order theories. Thus, the general issue raised in this Note is the following:

> *What is genuine 'computation' and what is 'extraneous mathematics'*
> *in the logical analysis of programming and its semantics?*

If we can isolate the former component, many different technical results in the current literature might be separable into different computational content plus a repetition of essentially the same mathematical overhead. We do not offer any general solution to this question here, but we do advocate some general awareness of the phenomenon.

2   Arrow Logic in a Nutshell

The intuition behind Arrow Logic is as follows. Binary relations may be thought of as denoting sets of arrows. Some key examples are 'arcs' in graphs, or 'transitions' for dynamic procedures in Computer Science, but one can also think of 'preferences' in the case of ranking relations (as found in current theories of reasoning in Artificial Intelligence, or social choice and economics). These arrows may have internal structure, whence they need not be identified with ordered pairs <source, target>: several arrows may share the same input-output pair, but also certain pairs may not be instantiated by an arrow. This motivates the following definitions (what follows here is a brief sketch: we refer to the references in the text for further technical details):

Arrow Frames   are tuples (A, C^3, R^2, I^1) with

A	a set of objects ('arrows') carrying three predicates:
C^3 x,yz	x is a 'composition' of y and z
R^2 x,y	y is a 'reversal' of x
I^1 x	x is an 'identity' arrow

Arrow Models M add a propositional valuation V here, and one can then interpret an appropriate modal propositional language expressing properties of (sets of) arrows using two modalities reflecting the basic 'ordering operations' of relational algebra:

M, x |= p	iff	x ∈ V(p)
M, x |= ¬ϕ	iff	*not* M, x |= ϕ
M, x |= ϕ&ψ	iff	M, x |= ϕ *and* M, x |= ψ
M, x |= ϕ•ψ	iff	*there exist* y, z *with* C x, yz *and* M, y |= ϕ , M, z |= ψ
M, x |= ϕ^v	iff	*there exists* y *with* R x, y *and* M, y |= ϕ
M, x |= Id	iff	I x

The minimal modal logic of this system is an obvious counterpart of its mono-modal predecessor, whose key principles are axioms of Modal Distribution:

$(\phi_1 \vee \phi_2)$ • ψ	$\leftrightarrow$	$(\phi_1•\psi) \vee (\phi_2•\psi)$
ϕ • $(\psi_1 \vee \psi_2)$	$\leftrightarrow$	$(\phi•\psi_1) \vee (\phi•\psi_2)$
$(\phi_1 \vee \phi_2)^v$	$\leftrightarrow$	$\phi_1^v \vee \phi_2^v$

A completeness theorem is provable here along standard lines, using Henkin models. (This minimal logic includes all the usual laws of Boolean Algebra.)

Next, one can add further axiomatic principles (taking cues from relational algebra) and analyze what constraint these impose on arrow frames via the usual semantic correspondences. In particular, we have that

(1) $\neg(\phi)^{\vee} \to (\neg\phi)^{\vee}$ iff $\forall x \exists y\ R\ x, y$

(2) $(\neg\phi)^{\vee} \to \neg(\phi)^{\vee}$ iff $\forall xyz: (R\ x, y\ \&\ R\ x, z) \to y=z$

Together, these make the binary relation R into a unary function r of 'reversal'. Then the 'double conversion' axiom makes the function r idempotent:

(3) $(\phi)^{\vee\vee} \leftrightarrow \phi$ iff $\forall x\ r(r(x)) = x$.

Let us assume this much henceforth in our arrow frames. Next, the following principles of Relational Algebra regulate the interaction of reversal and composition:

(4) $(\phi{\bullet}\psi)^{\vee} \to \psi^{\vee}{\bullet}\phi^{\vee}$ iff $\forall xyz:\ C\ x, yz \to C\ r(x), r(z)r(y)$

(5) $\phi \bullet \neg(\phi^{\vee}{\bullet}\psi) \to \neg\psi$ iff $\forall xyz:\ C\ x, yz \to C\ z, r(y)x$

Together (2), (4), (5) imply the further interchange law $\forall xyz:\ C\ x, yz \to C\ y, xr(z)$. Moreover, there is actually a more elegant form of axiom (5) without negation:

   $\phi\ \&\ (\psi{\bullet}\chi) \to \psi \bullet (\chi\ \&\ (\psi^{\vee}{\bullet}\phi))$.

Finally, the propositional constant Id will be involved in correspondences like

(6) $Id \to Id^{\vee}$ iff $\forall x:\ I\ x \to I\ r(x)$

(7) $Id{\bullet}\phi \to \phi$ iff $\forall xyz: (I\ y\ \&\ C\ x, yz) \to x=z$.

Obviously, there are many further choices here, and 'Arrow Logic' really stands for a family of modal logics, whose selection may depend on intended applications. Nevertheless, what might be the most natural 'computational core' in this field? Our recommendation would be as follows:

> *Universal Frame Constraints*
> Take only those principles concerning composition, converse and identity
> on arrow frames which lack existential import: i.e., their corresponding
> constraints can be formulated by purely *universal* first-order sentences.

One potential exception to this proposal is Associativity for composition:

(8a) $(\phi{\bullet}\psi){\bullet}\chi \to \phi{\bullet}(\psi{\bullet}\chi)$ iff $\forall xyzuv: (\ C\ x, yz\ \&\ C\ y, uv) \to$
                    $\exists w: (C\ x, uw\ \&\ C\ w, vz)\ )$

(8b) and likewise in the opposite direction.

Associativity is often tacitly presupposed in the formulation of dynamic semantics. In what follows we shall avoid this practice.

Further information about the landscape of systems for Arrow Logic may be found in van Benthem 1991, Marx, Németi & Sain 1992, Vakarelov 1992 and Venema 1992 (cf. also Appendix 1). Two further technical points deserve mention here. One is the existence of a certain uniformity. The above correspondences all follow from a general result in Modal Logic called the *Sahlqvist Theorem* (van Benthem 1984), which supplies an algorithm for computing frame conditions that can also be applied to other proposed candidates for inclusion in our core set. The other point is that the present modal language also has clear limits to its expressive power. Notably, one cannot force composition to become a partial function (general arrow logic allows more than one way of composing two transitions). For the latter purpose, *enriched modal formalisms* will be needed, employing further modal operators (cf. De Rijke 1992B, Roorda 1992), which we shall not pursue here. Of course, by the time we have enforced full representability of arrow frames via sets of ordered pairs, the resulting modal logic will be just as complex as the ordinary theory of representable relational algebras. The art is to know when to stop.

3 A Complete System of Dynamic Arrow Logic

Now, *Dynamic Arrow Logic* adds one infinitary operator to the above language:

$$M, x \models \phi^* \quad \text{iff} \quad x \; \textit{can be} \; \text{C-}\textit{decomposed into some finite}$$
$$\textit{sequence of arrows satisfying} \; \phi \; \textit{in} \; M \, .$$

What this says is that there exists some finite sequence of ϕ-arrows in M which allows of at least one way of successive composition via intermediate arrows so as to arrive at x . (Without Associativity, this does not imply that x could be obtained by any route of combinations from these same arrows.) Intuitively, ϕ^* describes the transitive closure of ϕ. It satisfies the following simple and natural principles:

(9)	axiom	$\phi \rightarrow \phi^*$
(10)	axiom	$\phi^* \bullet \phi^* \rightarrow \phi^*$
(11)	rule	*if* $\phi \rightarrow \alpha$ *and* $\alpha \bullet \alpha \rightarrow \alpha$ *are provable*, *then so is* $\phi^* \rightarrow \alpha$

These principles may be added to the earlier minimal arrow logic, to obtain a simple base system, but our preferred choice will consist of this minimal basis plus the earlier principles (1)–(5), to obtain a suitable axiomatic Dynamic Arrow Logic DAL. Here is an illustration of how this system works.

Example Derivation of Monotonicity for Iteration

If $\;\models \alpha \rightarrow \beta$ then $\;\models \alpha \rightarrow \beta^*$ (axil–m (9))
Alsl– $\;\models \beta^* \bullet \beta^* \rightarrow \beta^*$ (axil–m (10)) whence $\;\models \alpha^* \rightarrow \beta^*$ (rule (11)) ∎

Example   Derivation of Interchange for Iteration and Converse

i	$\phi \to \phi^*$	axiom (9)
ii	$\phi^{\vee} \to \phi^{*\vee}$	i plus monotonicity for converse
		(the latter follows from its distributivity)
iii	$(\phi^{*\vee} \bullet \phi^{*\vee}) \to (\phi^* \bullet \phi^*)^{\vee}$	this may be derived using axioms (3) and (4)
iv	$\phi^* \bullet \phi^* \to \phi^*$	axiom (10)
v	$(\phi^* \bullet \phi^*)^{\vee} \to \phi^{*\vee}$	iv plus monotonicity for converse
vi	$(\phi^{*\vee} \bullet \phi^{*\vee}) \to \phi^{*\vee}$	iii, v
vii	$\phi^{\vee *} \to \phi^{*\vee}$	ii, vi plus rule (11)
viii	$\phi^{\vee\vee *} \to \phi^{\vee *\vee}$	by similar reasoning
ix	$\phi \to \phi^{\vee\vee}$	axiom (3)
x	$\phi^* \to \phi^{\vee\vee *}$	monotonicity for iteration
xi	$\phi^* \to \phi^{\vee *\vee}$	x, viii
xii	$\phi^{*\vee} \to \phi^{\vee *\vee\vee}$	xi plus monotonicity for converse
xiii	$\phi^{\vee *\vee\vee} \to \phi^{\vee *}$	axiom (2)
xiv	$\phi^{*\vee} \to \phi^{\vee *}$	xii, xiii ∎

Completeness may be established for DAL , as well as several of its variants.

Theorem   DAL is complete for its intended interpretation.

Proof   Take some finite universe of relevant formulas which is closed under subformulas and which satisfies the following closure condition:

if ϕ^* is included,   then so is $\phi^* \bullet \phi^*$

Now consider the usual model of all maximally consistent sets in this restricted universe, setting (for all 'relevant' formulas):

C x, yz   iff $\forall \phi \in$ y, $\psi \in$ z: $\phi \bullet \psi \in$ x

R x, y   iff $\forall \phi \in$ y: $\phi^{\vee} \in$ x

Here we can prove the usual 'decompositions' for maximally consistent sets, such as

$\phi \bullet \psi \in$ x   iff   there exist y, z with C x, yz and $\phi \in$ y, $\psi \in$ z

using the minimal distribution axioms only. The key new case here is the following:

Claim $\phi^* \in$ x iff   some finite sequence of maximally consistent sets
                each containing ϕ 'C–composes' to x in the earlier sense.

Proof   From right to left. This is a straightforward induction on the length of the decomposition, using axioms (9), (10) and the closure condition on relevant formulas.

From left to right.   Describe the finite set of all 'finitely C–decomposable' maximally consistent sets in the usual way by means of one formula α , being the disjunction of all their conjoined 'complete descriptions' δ . Then we have

$$\vdash \phi \rightarrow \alpha :$$

since ϕ is provably equivalent to $\pounds_{\phi \in \delta} \delta$ in propositional logic, and α contains all these δ by definition. Next, we have

$$\vdash \alpha \bullet \alpha \rightarrow \alpha :$$

To see this, suppose that $(\alpha \bullet \alpha) \,\&\, \neg\alpha$ were consistent. Using Distributivity with respect to successive relevant formulas, $(\delta_1 \bullet \delta_2) \,\&\, \neg\alpha$ must be consistent for some maximally consistent δ_1 , δ_2 . And likewise, $(\delta_1 \bullet \delta_2) \,\&\, \neg\alpha \,\&\, \delta_3$ will be consistent for some maximally consistent δ_3 . Now, δ_1 , δ_2 must be in α , and moreover C $\delta_3, \delta_1 \delta_2$ by the definition of C and some deductive reasoning. Therefore, δ_3 is in α too (by definition), contradicting the consistency of $\neg\alpha \,\&\, \delta_3$. So, applying the iteration rule (11), we have

$$\vdash \phi^* \rightarrow \alpha$$

Therefore, if $\phi^* \in$ x , then x belongs to α .

Semantic evaluation in the canonical model will now proceed in harmony with the above syntactic decomposition: any relevant formula is true 'at' a maximally consistent set iff it belongs to that set. This completes our analysis of the basic case.

In order to deal with the additional axioms (1)–(5), their frame properties must be enforced in our finite canonical model. This may be done as follows:

i   one closes the universe of relevant formulas under Boolean operations and
          converses: the resulting infinite set of formulas will remain *logically finite*,
          given the Boolean laws and the interchange principles for converse,
ii   the definition of the relation C is to be modified by adding suitable clauses,
          so as to 'build in' the required additional frame properties.

First, the required behaviour of reversal is easy to obtain. One may define r(x) to be the maximally consistent set consisting of (all representatives of) $\{ \phi^v \mid \phi \in$ x $\}$: the available axioms make this an idempotent function inside the universe of relevant maximally consistent sets. For a more difficult case, consider axiom (5) with corresponding frame condition $\forall xyz: C\ x, yz \rightarrow C\ z, r(y)x$. One redefines:

$$C\ x, yz \qquad \text{iff} \qquad \text{both} \quad \forall \phi \in y,\ \psi \in z:\ \phi \bullet \psi \in x$$
$$\text{and} \quad \forall \phi v \in y,\ \psi \in x:\ \phi \bullet \psi \in z$$

This has been designed so as to validate the given frame condition. But now, we need to check that the earlier decomposition facts concerning maximally consistent sets are still available, to retain the harmony between membership and truth at such sets. Here are the two key cases:

$\phi \bullet \psi \in x$ iff   there exist y, z with $C\ x, yz$ and $\phi \in y$, $\psi \in z$

$\phi^* \in x$ iff   some finite sequence of maximally consistent sets containing ϕ 'C–composes' to x .

The crucial direction here is from left to right: can we find maximally consistent sets as required with C satisfying the additional condition? What we need is this. In the earlier proof, the sets y, z were constructed 'globally', by showing how successive selection yields a consistent set of formulas $x, \textbf{\&}y \bullet \textbf{\&}z$ with $\phi \in y$, $\psi \in z$. (For then, whenever $\alpha \bullet \beta$ is a relevant formula with $\alpha \in y$, $\beta \in z$, $\alpha \bullet \beta$ must belong to x, on pain of inconsistency.) Now, it suffices to show that, in this same situation, the set $z, \textbf{\&}r(y) \bullet \textbf{\&}x$ is consistent too. Here, we use a rule derived from axioms (2), (5):

if $\vdash \phi \to \neg(\psi \bullet \chi)$ *then* $\vdash \chi \to \neg(\psi^{\vee} \bullet \phi)$:

Then, if $\vdash z \to \neg (\textbf{\&}r(y) \bullet \textbf{\&}x)$, then $\vdash x \to \neg (\textbf{\&}rr(y) \bullet \textbf{\&}z)$, and hence also $\vdash x \to \neg (\textbf{\&}y \bullet \textbf{\&}z)$: contradicting the consistency of x , $\textbf{\&}y \bullet \textbf{\&}z$. The argument for iteration is similar. Moreover, the general case with all frame conditions implanted simultaneously employs the same reasoning. ∎

Corollary   DAL is decidable.

Proof   The preceding argument establishes not just axiomatic completeness but also the Finite Model Property. ∎

The above strategy for accommodating the relevant additional frame properties in the finite counter-model is that of Roorda 1991. More generally, we conjecture that every modal logic which is complete with respect to some finite set of Horn clause frame conditions has the Finite Model Property. But, will decidability will go through if the further existential property of Associativity is included in our basic arrow logic? This is more difficult, since the required additional worlds, whose existence is easily shown in a full Henkin model by traditional arguments, seem to over-flow the finite universe of 'relevant' maximally consistent sets during filtration. Dimiter Vakarelov has announced a proof, but the negative results in Andréka 1991 counsel caution.

Another strategy uses a *labeled version* of arrow logic with statements of the form '*arrow : assertion*' , which transcribe the above truth definition into a simple fragment of predicate logic. Labels might be either bare arrows (with relations of composition and converse), or complex descriptions from some (semi-)group. Following Roorda

1992, decidability might then be proved via an effective equivalence with some cut-free labeled sequent calculus for Arrow Logic, whose rules might use a format like:

Σ , x : A , y : B $\vert$– Δ *implies* Σ , xy : A•B $\vert$– Δ

Σ $\vert$– Δ , r(x) : A   *implies* Σ $\vert$– Δ , x : A^v

A general labeled approach to Arrow Logic must be left to further investigation.

4   Propositional Dynamic Logic with Arrows

Now what about a *Propositional Dynamic Logic* based on the above? The usual account in the literature considers the addition of a propositional component referring to truth at states essential, as it allows us some negation at least at the latter level. Since this is no longer true now, having this second component becomes more of a convenience. Nevertheless, we do think the resulting two-level system is a natural one: 'arrow talk' and 'state talk' belong together in an analysis of computation and general action. So as usual, add a Boolean propositional language, plus two mechanisms of interaction between the two resulting components:

a *test* 'mode' **?** taking statements to programs

a *domain* 'projection' **< >** taking programs to statements.

For notational convenience, we shall reserve ϕ, ψ , ... henceforth for state assertions and π, π_1, π_2, ... for describing programs in this two-tier system.

In line with the general modal analysis of the above, let us view this system with some greater abstraction. What we have is a *two-sorted modal logic*, whose models have both 'states' and 'arrows', and whose formulas are marked for intended interpretation at one of these. Both the arrow and state domains may carry internal structure, reflected in certain modalities, such as the earlier • and ^{v} referring to arrows. (States might be ordered by 'precedence' or 'preference' with appropriate modalities.) Our key point, however, is this. Even the modes and projections themselves may be viewed as 'non-homogeneous' modalities, reflecting certain structure correlating the two kinds of object in our models. For instance, 'test' is again a *distributive* modality, and so is 'domain':

$(\phi \vee \psi)$ **?** $\leftrightarrow$ ϕ **?** $\vee$ ψ**?**

<$\pi_1 \vee \pi_2$**>** $\leftrightarrow$ **<**π_1**>** $\vee$ **<**π_2**>**

whose interpretations run as follows:

M, x $\vert$= ϕ **?**   iff   *there exists some* s *with* T x, s *and* M, s $\vert$= ϕ

M, x $\vert$= **<**π**>**   iff   *there exists some* x *with* D s, x *and* M, x $\vert$= π

Intuitively, the first relation T x, s says that x is an identity arrow for the point s , while the second relation D s, x says that s is a left end-point of the arrow x .

Via the usual correspondences, further axioms on **?** , **< >** will then impose additional connections in our arrow frames between T and D .

Example   Connecting Identity Arrows and End-Points
The principle **<ϕ?> ↔ ϕ** (itself again a formula which is in modal 'Sahlqvist form') expresses the conjunction of

> ∀s ∃x : D s, x & Tx, s
>
> ∀sx: D s, x → ∀ s': T x, s' → s=s' .   ■

Also, axiomatic completeness proofs are straightforward here, with two kinds of maximally consistent sets: one for arrows and one for points. Thus everything about Propositional Dynamic Logic is Modal Logic: not just its two separate components, but also their connections. Further elegance may be achieved here by a reformulation. The following observation is made in van Benthem 1991:

Fact   There is one projection which is a Boolean homomorphism,
       namely the diagonal function λR• λx• Rxx

       There are exactly two homomorphic modes,
       namely λP• λxy• Px   and λP• λxy• Py .

Thus, we can introduce three matching modalities with corresponding new binary relations in their semantics:

> M, s |= Dπ   iff   *for some* x , Δ s, x *and* M, x |= π
>
> M, x |= Lϕ   iff   *for some* s , L s, x *and* M, s |= ϕ
>
> M, x |= Rϕ   iff   *for some* s , R s, x *and* M, s |= ϕ

These modalities satisfy not just the Distribution axioms, but they also commute with Boolean negation (just like relational converse), so that we can take Δ , L , R to be *functions*. This set-up is more elegant, as well as easy to use. (It may still be simplified a bit by dropping Rϕ in favour of (Lϕ)^v.) For instance, one source of axiomatic principles is the interaction of various operators:

Observation

DLϕ ↔ ϕ	expresses that	∀s: L Δ (s) = s
LDπ ↔ (π&Id) • T	expresses that	∀x ∃y : C x, Δ L (x) y
Lϕ•π → Lϕ	expresses that	∀xyz: C x, yz → L(x) = L(y) .

One may achieve exactly the power of the standard system with these new primitives under the following

> *Translation from old to new format*
>
> ϕ ? :   L ϕ & Id
>
> < π > :   D (π•T)

Analyzing the usual axioms of Propositional Dynamic Logic in this fashion is a straightforward exercise. We list the key principles that turn out to be needed (these allow us to represent statements $<\pi> \phi$ faithfully as $D ((\pi \And R\phi) \bullet T)$:

1	$D \pi \rightarrow D (\pi \And Id)$
2	$Id \rightarrow (L\phi \leftrightarrow R\phi)$
3a	$DL\phi \leftrightarrow \phi$
3b	$DR\phi \leftrightarrow \phi$
4	$\pi_1 \And RD\pi_2 \leftrightarrow \pi_1 \bullet (\pi_2 \And Id)$
5	$(\pi_1 \bullet \pi_2) \And R\phi \leftrightarrow \pi_1 \bullet (\pi_2 \And R\phi)$

Their corresponding frame conditions can be computed by hand, or again with a Sahlqvist algorithm, as they are all of the appropriate modal form. These principles suffice for deriving various other useful ones, such as the reductions

$$(Id \And L\phi) \bullet \pi \leftrightarrow L\phi \And \pi$$
$$\pi \bullet (Id \And R\phi) \leftrightarrow \pi \And R\phi .$$

Finally, there is also a converse route, via two more schemata:

Translation from old to new format

$L\phi$: $\phi ? \bullet T$

$D\pi$: $< Id \And \pi >$

The same style of analysis may be applied to richer systems of dynamic logic, having additional structure in their state domains (cf. van Benthem 1991, de Rijke 1992A). One example is the 'dynamic modal logic' in De Rijke's contribution to this Volume, which features modes over information states with an inclusion order Ó . This may be treated by introducing another propositional constant at the arrow level, say, E for 'inclusion' (perhaps with suitable axioms expressing its transitivity and reflexivity). Then, the logic of updating and revision will employ special defined arrows, such as

$E \And R\phi$	update transition for ϕ
$(E \And R\phi) \And \neg((E \And R\phi) \bullet (E \And \neg Id))$	minimal update transition for ϕ

This may provide a workable alternative where the undecidability of the full system is circumvented. Roughly speaking, the arrow version should stay on the right side of the '2D-boundary' which allows embedding of two-dimensional grids in the models, and hence encoding of full Turing machine computation.

Acknowledgement

5   *Appendix 1*   **From Amsterdam to Budapest**

Arrow Logic in its 'Amsterdam manifestation' says that dynamic transitions need not be identified with the ordered pairs over some underlying state set. This idea has really two different aspects. Distinct arrows may correspond to the same pair of <input, output> , but also, not every such pair need correspond to an available arrow. This shows very well in the following less standard example:

> Let arrows be *functions* $f : A \rightarrow B$ giving rise to, but not identifiable with, ordered pairs <A, B> of 'source' and 'target'. Then, the relation C expresses the partial function of composition of mappings, while the reversal relation R will hold between a function and its inverse, if available.

This model will validate all of the earlier core principles, at least, in their appropriate versions after functionality for reversal has been dropped. For instance, axiom (5) now expresses the fact that, whenever $f = g_0 h$ and $k = g^{-1}$, then also $h = k_0 f$.

Nevertheless, there is also an interesting more 'conservative' variant found in various earlier and recent publications from Budapest, where arrows are still ordered pairs, but one merely gives up the idea that all ordered pairs are available as arrows. Essentially this takes us to a universally first-order definable class of arrow frames which can be represented via sets of ordered pairs (though not necessarily full Cartesian products). Its complete logic can be determined in our formalism, and it turns out to be decidable as well (Marx Németi & Sain 1992). This system is another natural, richer stopping point in the arrow landscape, including the earlier systems presented in Section 2 above, with additional axioms expressing essentially the uniqueness of the pair of identity arrows surrounding an arbitrary arrow, as well as their 'proper fit' with composition and reversal. Various weaker natural arrow logics with desirable meta-properties (decidability, interpolation, etcetera) may be found in Németi 1987 (see also the survey Németi 1991 for more extensive documentation). Simón 1992 investigates deduction theorems for arrow logics, showing that our basic systems lacks one. Finally, Andréka 1991 provides a method for proving results on non-finite-axiomatizability in the presence of full Associativity. Even undecidability lies around the corner, in this perspective, as soon as one acquires enough power to perform the usual encoding of relation-algebraic quasi-equations into equations.

6   *Appendix 2*   **Dynamic Arrow Logic with a Fixed-Point Operator**

The analysis of this Note may be extended to prove completeness for a more powerful system of Dynamic Arrow Logic which has the well-known minimal fixed-point operator $\mu p \bullet \phi(p)$. Its two key derivation rules are as follows:

$$if \;\; |\!\!- \phi(\alpha) \to \alpha \qquad\qquad then \;\; |\!\!- \mu p \bullet \phi(p) \to \alpha \qquad\qquad\qquad\qquad I$$

$$if \;\; |\!\!- \beta \to \mu p \bullet \phi(p) \quad then \;\; |\!\!- \phi(\beta) \to \mu p \bullet \phi(p) \qquad\qquad\qquad II$$

This language defines iterations ϕ^* in our sense via the fixed-point formula

$$\mu p \bullet \; \phi \vee p \bullet p \; .$$

(Its successive approximations give us all C-combinations that were involved in the earlier semantic definition.) The derivation rules for iteration then become derivable from the above two rules: I corresponds to rule (11), while II gives the effect of the axioms (9), (10). In the completeness theorem, these allow us to generalize the earlier argument for the crucial decomposition:

$$\mu p \bullet \phi(p) \in \; x \quad\quad iff \quad\quad x \; \textit{belongs to some finite iteration of the operator}$$
$$\lambda p \bullet \phi(p) \; \textit{starting from the empty set for} \;\; p \; .$$

7 *Appendix 3* **Connections with Categorial Logic and Action Algebra**

Dynamic Arrow Logic may also be compared to a dynamic version of categorial logic, as employed in current categorial grammars, extended with Kleene iteration. At base level, this connection runs between ordinary arrow logic and standard systems such as the *Lambek Calculus* with two directed functional slashes (cf. van Benthem 1991, 1992 for details):

$$a \backslash b \quad := \quad \neg (a^\vee \bullet \neg b)$$
$$b/a \quad := \quad \neg (\neg b \bullet a^\vee)$$

Moreover, categorial product goes to composition $\bullet$. The two basic categorial laws then express the basic interaction principles for C and r on arrow frames:

$$a \bullet (a \backslash b) \; \leq \; b \qquad\qquad\qquad \forall xyz: \; C \, x, yz \to C \, z, r(y)x$$
$$(b/a) \bullet a \; \leq b \qquad\qquad\qquad \forall xyz: \; C \, x, yz \to C \, y, xr(z) \; .$$

This gives us the two implications

$$X \leq a \backslash b \;\; \Rightarrow \;\; a \bullet X \leq b \qquad\qquad X \leq b/a \;\; \Rightarrow \;\; X \bullet a \leq b$$

Their converses (which generate all of the Lambek Calculus) require no more. For instance, suppose that $a \bullet X \leq b$. Now, $X \, \& \, (a^\vee \bullet \neg b) \leq a^\vee \bullet (\neg b \, \& \, (a \bullet X))$ by the first interaction principle, whence we obtain $X \, \& \, (a^\vee \bullet \neg b) \leq a^\vee \bullet (\neg b \, \& \, b)$ and then $X \, \& \, (a^\vee \bullet \neg b) \leq a^\vee \bullet 0 \leq 0$. I.e., $X \leq \neg (a^\vee \bullet \neg b)$.

Thus, Basic Arrow Logic contains the Lambek Calculus, and it even does so faithfully, thanks to the completeness theorem in Mikulás 1992. Many further connections between categorial logics and arrow logics remain to be explored.

With * added, we get some obvious further principles, such as

$$(a\backslash a)^* \;=\; (a\backslash a) \,,$$

due to Tarski & Ng. Note thow this may be derived in Dynamic Arrow Logic:

1 $(a\backslash a) \;\leq\; (a\backslash a)^*$ axiom 9

2 $(a\backslash a) \;\leq\; (a\backslash a)$

   $(a\backslash a) \bullet (a\backslash a) \;\leq\; (a\backslash a)$ by the above categorial rules (derivable in arrow logic)

   $(a\backslash a)^* \;\leq\; (a\backslash a)$ by rule (11)

A related system is the *Action Algebra* of Pratt 1992 and previous publications, which may be viewed as a standard categorial logic enriched with iteration and disjunction. It would be of interest to determine the precise connection with arrow logic here. What is easy to determine, at least, is the following 'arrow content' of the equational axiomatization offered by Pratt. Its basic axioms each exemplify one of four kinds of assertion in our framework:

i   consequences of the minimal arrow logic – in particular, the basic laws of monotonicity (a typical example is $\vdash a \rightarrow b \;\leq\; a \rightarrow (b + b')$ ")

ii expressions of categorial principles, whose content was the basic interaction between composition and converse (as expressed in the inequalities $\vdash a(a \rightarrow b) \;\leq\; b \;\leq\; a \rightarrow ab$ ")

iii universally valid principles for iteration, such as its monotonicity (compare $\vdash a^* \;\leq\; (a+b)^*$ ")

iv associativity for composition (whose precise strength remains to be determined in the arrow framework, as we have seen).

8   *Appendix 4*   **Predicate Arrow Logic**

It may also be of interest to ask whether the above style of analysis applies to ordinary predicate logic. In particular, does its undecidability go away too, once we give up the usual bias toward ordered pairs? First, the formulation is easy:

> Take a two-sorted language with 'objects' and 'arrows'
> and read, say, "Rxy" as $\exists a (\, Ra \;\&\; l(a) = x \;\&\; r(a) = y \,)$.

Thus we need unary predicates for the old relations, plus two new auxiliary cross-sorted maps l , r identifying end-points of arrows. (For general n-any relations, we may need a more-dimensional version of Arrow Logic, as in Vakarelov 1992.) But the resulting system still faithfully embeds ordinary predicate logic, and hence it is at least as complex.

Question   What would have to be weakened in standard predicate logic to get an arrow-based decidable version, either in the Amsterdam or the Budapest Way?

What this analysis shows is that versions of Arrow Logic can also get undecidable without identifying arrows with ordered pairs, viz. by putting in additional expressive power via modal operators reflecting further predicate-logical types of statement, such as a 'universal modality' or a 'difference operator' or yet other additions (cf. again De Rijke 1992B, Roorda 1992 – as well as the various recent publications by Gargov, Goranko, Passy and others from the 'Sofia School' in enriched modal logic).

Question   What happens to the previous versions of Propositional Arrow Logic if one adds a 'universal modality' or a 'difference operator', or yet other notions from extended Modal Logic?

A good concrete example here is the traditional formula enforcing infinity of a binary relation in its models:

$$\forall x \, \neg Rxx \, \& \quad \forall x \, \exists y \, Rxy \, \& \quad \forall xy \, (Rxy \, \rightarrow \forall z \, (Ryz \rightarrow Rxz)) \, .$$

Its 'arrow transcription' reflects our natural reasoning about this formula, in terms of growing chains in arrow diagrams. Analyzing the usual argument about its models, one finds how little is needed to show their infinity, thus destroying the Finite Model Property and endangering Decidability.

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