What One May Come to Know

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1 Verificationism and the Fitch Paradox

The general verificationist thesis says that *what is true can be known* – or formally:

$$\phi \rightarrow \Diamond K\phi \qquad \textbf{\textit{VT}}$$

A surprising and much-discussed argument by Fitch trivializes this principle. It uses just a weak modal epistemic logic to show that ***VT*** collapses the notions of truth and knowledge, by taking the following clever substitution instance for ϕ:

$$P \wedge \neg KP \rightarrow \Diamond K(P \wedge \neg KP)$$

Then we have the following chain of three conditionals:

(a) $\Diamond K(P \wedge \neg KP) \rightarrow \Diamond (KP \wedge K\neg KP)$

in the minimal modal logic for the knowledge operator *K*,
(b) $\Diamond (KP \wedge K\neg KP) \rightarrow \Diamond (KP \wedge \neg KP)$ in the modal logic *T*,

and so (c) $\Diamond (KP \wedge \neg KP) \rightarrow \bot$ in the minimal modal logic for <>.

Thus, a contradiction has been derived from the assumption $P \wedge \neg KP$, and we have shown over-all that *P* implies *KP*, making truth and knowledge equivalent.

Proposed remedies for the Paradox fall mainly into two kinds (cf. Brogaard and Salerno 2002, Wansing 2002). Some weaken the logic in the argument still further. This is like tuning down the volume on your radio so as not to hear the bad news. You will not hear much good news either. Other remedies leave the logic untouched, but weaken the verificationist principle itself. This is like censoring the news: you hear things loud and clear, but they may not be so interesting. The proposal made below falls into the latter category, but using a different perspective from mere tinkering with proof rules or premises. We will emphasize positive reasons why ***VT*** can, and sometimes should fail, having to do with the ways in which we learn new information.

2 Knowable propositions and learning

Fitch's substitution instance exemplifies a much older problem about knowledge called *Moore's Paradox*. It consists in the observation that the statement

"*P*, but I don't know it"

can be true, but cannot be known, as *K(P & ¬KP)* evidently implies a contradiction. [1]

Now, in an epistemic logic for a single agent, the possible knowledge of a proposition ϕ requires that $K\phi$ be satisfiable at some world in some model, and hence in all alternatives to that world. This differs from ordinary epistemic satisfiability, which just demands truth of ϕ at some world in some model. Tennant 2002 argues persuasively for the following restriction on the intended applications of ***VT*** to propositions ϕ:

$K\phi$ is consistent **CK**

In simple epistemic *S5*-models, this special requirement amounts to *global satisfiability* of ϕ: i.e., its truth throughout at least one model. Like ordinary satisfiability, this notion is decidable for most modal logics [2] (Note 1), and hence constraints of knowability can be formulated at least in an effective manner. But there is a bit more to the situation! Our first observation is that ***CK*** only partially captures the intuition behind ***VT***.

3 A dynamic shift: consistent update

Consider any epistemic model *(**M**, s)*, with a designated world *s* standing for the actual situation. What might be known in this setting seems restricted, intuitively, to what might be known correctly *about that actual situation*. We know already that it is one of the worlds in ***M***. What we might learn is that this model can be shrunk still further, zooming in on the location of *s*. In this dynamic sense, the verificationist principle that every true statement may be known amounts to stating that

What is true may come to be *known* $\boldsymbol{VT^*}$

Clearly, $\boldsymbol{VT^*}$ only holds for propositions ϕ that satisfy ***CK***. But it is more demanding. We need truth of $K\phi$ not in just any model, but *in some submodel of the current one*.

[1] The better-known version of Moore's Paradox rather has the doxastic form "P, but I don't believe it". But Moore 1962 does discuss a knowledge version as well.

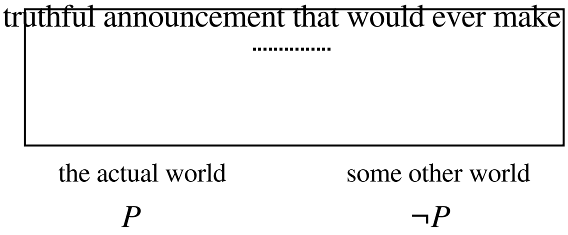
[2] Still, the computational complexity of global decidability may go up from that for standard *S5*, as we are now adding a so-called 'universal modality'. Cf. Blackburn, de Rijke & Venema 2001.

Fact ***CK*** does not imply $\boldsymbol{VT}^*$ for all propositions ϕ.

Proof Let $<>\phi$ be the existential dual of the operator K, standing for the epistemic (not the earlier modal!) notion of 'holding it possible that ϕ'. Now consider the statement

$$\phi = (P \,\&\, <>\neg P) \vee K\neg P$$

This is knowable in the sense of ***CK***, since $K((P \,\&\, <>\neg P) \vee K\neg P)$ is consistent: it holds in a model consisting of just one world with $\neg P$, where the agent knows that $\neg P$. But here is a two-world *S5*-model $\boldsymbol{M}$ where ϕ holds in the actual world, even though there is no truthful announcement that would ever make us learn that ϕ:



In the actual world, $(P \,\&\, <>\neg P) \vee K\neg P$ holds, but it fails in the other world. Hence, $K((P \,\&\, <>\neg P) \vee K\neg P)$ fails in the actual world. The only truthful proper update of this model $\boldsymbol{M}$ would just retain its actual world. But in the resulting one-world epistemic model with the proposition letter P true, $K((P \,\&\, <>\neg P) \vee K\neg P)$ fails. **QED**

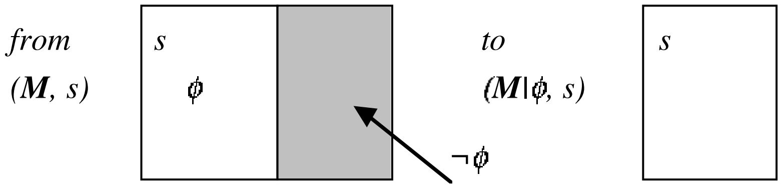
Thus, consistency of $K\phi$ need not be enough if we wish to learn that ϕ here and now in any model where it holds, as expressed by $\boldsymbol{VT}^*$. Our first point is then that

> *In a natural learning scenario, the Verificationist Thesis places stronger requirements on propositions than those stated so far in the literature.*

This observation suggests a closer look at the general dynamic viewpoint underpinning $\boldsymbol{VT}^*$. In a nutshell, what we know is the result of *actions of learning*.

4 Epistemic logic dynamified

The simplest way of learning is by being told through a true new proposition which prunes the current epistemic model. In particular, a *public announcement* $\phi!$ of assertion ϕ does not just evaluate ϕ truth-conditionally in the current model $(\boldsymbol{M}, s)$. It rather changes that model, by eliminating all those worlds from it which fail to satisfy ϕ:



This scenario works for simple questions and answers, but also for more intricate puzzles of knowledge and ignorance (van Benthem 2003) [3]. Thus, the semantic setting for basic forms of learning is a family of epistemic models standing for the relevant information states, related by a repertoire of actions of announcing propositions that increase information by moving from one model to another. Complete systems for this dynamified epistemics mix standard epistemic logics with dynamic logics of actions, with expressions describing what holds after an action was performed:

$[a]\psi$ ψ holds after every successful execution of action *a*

The expression *a* may be a computer program or some physical action, or a speech act. In particular, in this way, one can express and investigate systematic statements about epistemic effects of successful communication:

$[\phi!]K_j\psi$ after a true public announcement of ϕ, *j* knows that ψ

There are complete and decidable logical calculi for this richer language extending epistemic *S5* with suitable laws for actions. These include axioms relating statements about the result of an action to those that were true before. [4] In particular, epistemic logics for communication emphasize the multi-agent character of speakers, hearers, and audiences. Accordingly, the language can iterate knowledge assertions, as in

$K_1\neg K_2P$ *1* knows that *2* does not know that *P*

There are also new notions for groups of agents *G*, such as common knowledge

$C_G\phi$: everyone knows that ϕ, and they also know that the others know,

[3] More sophisticated 'product update' formats, beyond simple elimination, model complex forms of communication mixing public actions and information hiding. This more general format covers much of what happens in games and more realistic communicative settings.

[4] As an illustration, one typical valid principle reduces knowledge after communication to 'relativized knowledge' that must be true before it: $[A!]\, K_i\, \phi \leftrightarrow (A \rightarrow K_i\,(A \rightarrow [A!]\, \phi))$.

and so on to any finite depth of iteration of mutual knowledge operators.

This point about epistemic interaction will return below, as what seems like paradoxes for the case of lonesome knowers may look brighter in groups.

5 The dynamic logic of learning

One currently open issue in dynamic epistemic logics concerns the generic effects of public announcement. At first, this seems simple. Here is a putative, almost self-evident *Learning Principle* about the epistemic effects of a public statement that ϕ.

> Announcing ϕ publicly in a group G makes ϕ common knowledge:
> or in a dynamic-epistemic formula: $[\phi!]C_G\phi$ ***LP***

Indeed, ***LP*** holds for atomic statements ϕ and many other more complex formulas. But even so, it is false in general! E.g., if someone tells you truly

> "*P*, but you don't know it",

the result is a model where *P* holds everywhere, and your ignorance has disappeared. Of course, this is Moore's Paradox again, but now in a dynamic epistemic setting. This very update would occur in the model used earlier in Section 3 to strengthen the Consistency of Knowledge principle ***CK***. Combining this observation with our earlier ones in Sections 1,2, we arrive at the second main observation of this paper:

> *The 'paradoxical' behaviour of* ***VT*** *closely reflects that of* ***LP***.

But this analogy also suggests another way of looking at the Verificationist Thesis. Upon reflection, the Learning Principle just seems an overly hasty assertion, and the given counter-example seems very natural. Indeed, announcements of ignorance are not just philosophical conuncrums. They are made frequently, and they can be very useful.

E.g., in well-known puzzles like Muddy Children it is precisely public announcements of ignorance which drive the solution process toward common knowledge of the true state of affairs. In a simple case, the story runs as follows (cf. van Benthem 2002):

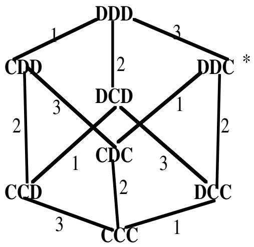
> *After playing outside, two of three children have mud on their foreheads. They all see the others, but not themselves, so they do not know their own status. Now their Father comes and says: "At least one of you is dirty". He then asks: "Does anyone know if he is dirty?" The children answer truthfully. As questions and answers repeat, what will happen?*

Nobody knows in the first round. But in the second round, each muddy child can figure out her status by reasoning as follows:

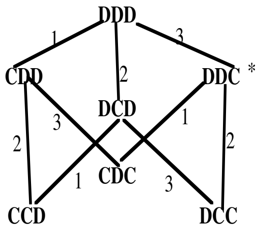
> "If I were clean, the one dirty child I see would have seen only clean children around her, and so she would have known that she was dirty at once. But she did not. So I must be dirty, too!"

In the relevant epistemic model, worlds assign ***D*** or ***C*** to each child. The actual world is ***DDC***. Moreover, a child knows about the others' faces, but not about his own. This is indicated by the labelled uncertainty lines in the following diagrams. Updates in ther sense of Section 4 would start with the Father's elimination of the world ***CCC***:

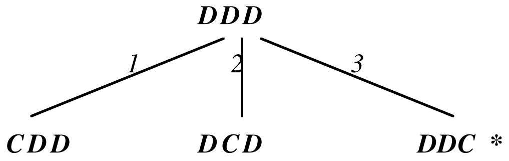
from



to



When no one knows his status, the bottom worlds disappear:



The final update is to ***DDC*** *

With *k* muddy children, *k* rounds of public ignorance assertions are needed to achieve common knowledge about who is dirty, while the announcement that the muddy children know their status achieves common knowledge of the whole situation. Thus, assertions of ignorance can drive a positive process of gathering information, and their ability to invalidate themselves may even be the crowning event. The last announcement of ignorance for the muddy children led to their knowing the actual world.

Logicians have adapted to this situation, and turned a problem into an object of study. What we see in puzzles like this is merely that communication is more interesting than what the putative principle *LP* suggests. For instance, we can investigate what special syntactic forms of assertion *do* become common knowledge upon announcement. [5] And

[5] For instance, all *universal* modal formulas are self-fulfilling. These are the ones constructed using atoms and their negations, conjunction, disjunction, K_i and C_G. But there are other self-fulfilling types

this again suggests more general classifications. Statements of atomic facts may be called *self-fulfilling*: once announced, their common knowledge results. Moore's statement, on the other hand, is *self-refuting*: once announced, its negation always becomes common knowledge. But there are also wavering statements in between. Given the analogy between ***VT*** and ***LP***, one might also develop an analogous enriched verificationist logic, distinguishing different roles for different types of statement.

In the remainder of this paper, we develop this technical theme a bit further. What does knowability or learnability of propositions amount to in a dynamic epistemic setting?

6 Exploring learnable propositions

As in the usual discussion of the Knower paradox, consider the case of a single agent. Suggestions for the case of more agents will follow later. Define a *learnable proposition* ϕ as one whose truth can always become known by announcement of some suitable true formula *A*. I.e., the following implication is valid for such formulas:

$$\phi \rightarrow \exists A <A!>K\phi \qquad \textit{Learnability}$$

Here the existential modality $<A!>K\phi$, dual to the above $[A!]K\phi$, says that a truthful announcement of *A* is possible in the current model $(\boldsymbol{M}, s)$, leading to knowledge of ϕ after the corresponding update. Note also that the consequent $\exists A <A!>K\phi$ is shorthand for an infinite disjunction over all formulas *A* of our language.

Fact Learnability is decidable in *S5*.

Proof It is well-known that all models of an *S5*-language for a finite set of proposition letters can be finitely enumerated, as the only options that matter to truth are which propositional valuations occur in the set of worlds. Thus, for each epistemic formula ϕ, we can enumerate all relevant models $\boldsymbol{M}, s \models \phi$ in a finite list. Now, for ϕ to be learnable in the above sense, each of the models $(\boldsymbol{M}, s)$ in that list must have a submodel $\boldsymbol{N}$ (again in the list) containing s with ϕ true in every world of that submodel. It is immediate that this can be checked effectively. **QED**

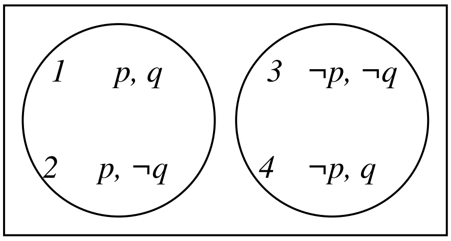
of statement as well, such as $<>p$. A complete syntactic characterization of the self-fulfilling patterns has been one of the open model-theoretic problems of epistemic dynamic logic since the mid 1990s.

A stricter form of learnability would demand more uniformly that there be some *finite* set of announcements *A* one of which must lead to knowledge of ϕ in any given model of ϕ. This learnability by finite cases is equivalent to the above version, however, by the compactness theorem for *S5* – or more simply, by the above enumeration argument. A truly stronger uniform version of the requirement is the existence of one single assertion *A* such that the following formula is valid:

$\phi \rightarrow <A!>K\phi$ *Uniform learnability*

Fact Uniform learnability is stronger than learnability.

Proof Consider the following 4-world epistemic *S5*-model ***M***:



Let ϕ be an epistemic formula which is only true in the following minimal situations:

(a) in the pictured *4*-world model ***M***: at the worlds *1, 3*, and no others

(b) in the two smaller models indicated by the ellipses: at both worlds.

It is easy to write down such a formula explicitly. [6] According to the above description (a), (b), ϕ is learnable: some update makes it known it whenever it is true. But is clear no single formula *A* does this job uniformly, since the selection of the submodel in ***M*** has to depend on which of the two ϕ–worlds is our point of departure. **QED**

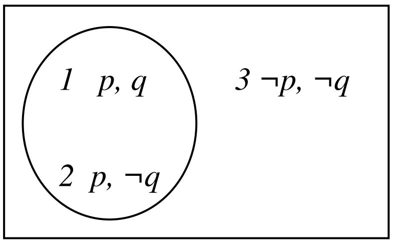
Still stronger is the case where announcing ϕ *itself* produces its knowledge. This is the earlier learning situation of self-fulfilling assertions, restricted to the single-agent case:

$\phi \rightarrow <\phi!>K\phi$ *Self-fulfillment*

[6] One describes a model ***M*** globally by stating the possibility of each valuation in it, viewed as a conjunction of atoms and their negations, and then taking a universal modality over the disjunction of all these. Statements picking out specific worlds *s* in the model can then be conjoined with this. The total formula is a disjunction of a number of such local descriptions for various *(**M**, s).*

Fact Statements can be uniformly learnable without being self-fulfilling.

Proof Consider the following model ***M***, in the same style as the preceding one:



Let ϕ hold only in

(a) in model ***M***: at world *1*,

(b) in ***M***'s oval two-world submodel: at both worlds.

Uniform learnability is satisfied. In every model *(**M**, s)* where this formula ϕ holds, announcing the true atomic fact *p* makes ϕ true throughout the resulting model. But self-fulfilment fails. Announcing the true statement ϕ *itself* in the *3*-world initial model *(**M**, 1)* would leave just the *1*-world submodel *p, q*. The reason is that, since we are not in the smaller submodel, the parts of the disjunction ϕ referring to the latter will be false. But in this one-world submodel *{1}*, ϕ fails by its definition. **QED**

The upshot of this more detailed analysis is as follows.***VP*** and ***LP*** are analogous to some extent, but the two putative learning principles do not coincide. On the way to this conclusion, we have seen something positive: the flexibility of the dynamic framework in phrasing different versions of learnability. This concludes our account of the single agent setting for epistemic update and learning. Our third main point, then, is that

> ***VT**, **VT*** and **LP** point toward an interesting general logic of knowledge assertions, announcements, and learning actions.*

Looking at some possible extensions adds still further detail to this perspective.

7 Refining the issues

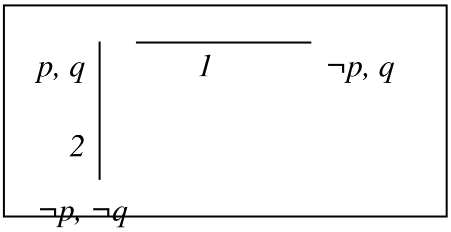
Our analysis has looked at the Verificationist Thesis and the Paradox of the Knower in terms of epistemic actions. This does not solve the original problem, but places it in a broader setting of interaction between many agents. In particular, the original Paradox of the Knower now becomes a special case in several senses. First, even with one agent, two different senses of learnability emerged: either by means of fixed assertions, or by context-dependent assertions. But also, the multi-agent setting suggests further refinements. With a single agent, the only candidate for the required knowledge level was $K\phi$. But now one can require knowledge for *other* agents as well: some, or all. E.g., Moore's Paradox disappears with some *other* agent *2* learning that

"*P*, and *1* does not know it"

as the iterated epistemic formula $K_2(P \ \& \ \neg K_1 P)$ may quite well become true. Also, with groups, we can strengthen the original knowledge condition of ***VT*** as follows:

If ϕ is true, then it is possible that ϕ becomes *common knowledge*.

Perhaps each member finds out part of the truth, and by pooling this information, they arrive at $C_G\phi$. E.g., consider the following model ***M*** with actual world *p, q*:



Announcing *q* will make *2* know the Moore statement that "*p* and *1* does not know it". But this can never become common knowledge in the group *{1, 2}*. What can become common knowledge, however, is *p & q*, when *1* announces that *q*, and *2* then says *p*.

Many further distinctions can be made in interactive versions of knowability. There may be a price for this expressive power, however, in that the above results about learnability may become harder to formulate and prove for many agents. [7] Also, more delicate learning scenarios involving secrecy, hiding, and even misleading, occur in epistemic update logic, with different subgroups getting different information about the facts: cf. Baltag, Moss & Solecki 1998. Finally, the dynamic component of the logic also adds a dimension. In a piece of recent jargon, the phrase "knowable" suffers from the common disease called '∃–sickness'. This means using an existentially quantified notion in a situation where more explicit information is available, whose logic could be brought out.

[7] Technically, a multi-agent epistemic language with a common knowledge modality is not like plain *S5*. Simple enumeration arguments like that for decidability of single-agent learnability no longer hold, and the same is true for other model-theoretic techniques. In particular, we do not know precisely how our earlier results of Section 6 on learnability fare in this setting.

Common symptoms are frequent uses of "-ility"'s. Compare: provability versus a concrete proof, past tense as 'once upon a time' versus some specific past episode, solvability versus producing an algorithm, winnability of a game versus a concrete winning strategy, etc. A full-fledged dynamic epistemic logic would cure the sickness in the particular case of 'knowability' by making learning actions and their properties an explicit part of the logic of verificationism, however construed. [8]

8 Conclusion

We have shown that knowability of a proposition involves more than consistency of its being known, by placing the Paradox of the Knower in a dynamic setting where learning involves changing the current epistemic model. The Verificationist Thesis then turns out related to the Learning Principle for public announcement. Elaborating this analogy, we found different versions of knowability in an update setting, plus interesting extensions to multi-agent learning. This twist in perspective also reflects a change in mood. Much of the literature on the Fitch Paradox seems concerned with averting a disaster, and saving as large a chunk of verificationism as possible from the clutches of inconsistency. In our perspective, there is no saving ***VT*** – but there is also no such gloom. For in losing a principle, we gain a general logical study of knowledge and learning actions, and their subtle properties. The failure of naïve verificationism just highlights the intriguing ways in which human communication works.

[8] As a side benefit, our proposal also enriches dynamic epistemic logic. Our observations about single-agent *S5* show that learnability assertions are definable there, and do not add anything new to the language. But now consider public announcements *A!* in a first-order language, where a formula $A = A(x)$ restricts the full domain to the definable subdomain of objects satisfying A. Now, expressive power may increase. E.g., take any strict linear order satisfying the first-order theory of $(N, <)$ which extends beyond N by adding copies of the integers Z. Its only first-order definable subsets of objects are the finite and the co-finite ones. Now consider the first-order learnability assertion that

> 'some true announcement makes the following true: the current object *n* is
> the greatest point, while every object different from zero has a predecessor'.

This can only be true for those objects *n* which lie at some finite distance from the zero of the model. These form an initial copy of *N*, which is not definable in first-order logic. Balder ten Cate has pointed out one might also do this argument in a temporal language, closer to the epistemic modal original.

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