2

Basic language and semantics

2.1 Syntax of modal propositional logic

A logical formalism starts with a language, a system of *patterns* behind some practice of communication and reasoning. These patterns are formal and austere, but that is precisely why they highlight basic features of the phenomenon described, while also suggesting analogies across different situations. Our basic language has the following syntax:

Definition 2.1.1 (Basic modal language)**.** Formulas are defined as follows. We first chose a basic set of unanalyzed propositions:

$$AT ::= p, q, r \ldots \text{plus } \top \text{ (“always true”) and } \bot \text{ (“always false”)}$$

Next, we define inductively how to construct further expressions, using the format:

$$\varphi ::= AT \mid \neg\varphi \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi) \mid (\varphi \rightarrow \psi) \mid \Diamond\varphi \mid \Box\varphi$$

Here is how one reads items on the last line: “all atoms are formulas”, “if φ is a formula, then so is $\neg\varphi$”, “if φ, ψ are formulas, then so is $(\varphi \wedge \psi)$”, etcetera. The understanding is that formulas are all and only the syntactic strings arising from this recursive process in a finite number of steps. While this format, originally invented for defining programming languages, is more terse than the usual formulations in most logic textbooks, it is very perspicuous – and its brevity in ink and paper also helps save the tropical rainforest.

Remark (Notation)**.** In many passages in these lectures, I will denote arbitrary propositions by proposition letters $p, q, \ldots$ but sometimes also by capital letters $A, B, \ldots$ or Greek symbols $\varphi, \psi, \ldots$. This practice is not very consistent, but most readers should agree that it is nice to have different clothes to wear, depending on one’s mood.

There are many possible readings for the modality $\Box$ (pronounced “box”), as we have hinted at in our Introduction: necessary truth,

knowledge, obligation, ... $\Diamond$ is the dual modality, called "diamond". Getting ahead of our formal presentation, we can think of this second notion as defined, since there are two intuitive "dualities" between the two modalities – like those between the universal quantifier $\forall$ and existential quantifier $\exists$ of first-order logic:

Fact. The following two principles are intuitively valid:

$$(a)\ \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi \qquad\qquad (b)\ \Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$$

Interestingly, many notions in natural language come in dual pairs: the universal and existential quantifiers, "always" and "sometimes", obligation and permission, "already" and "not yet", etcetera. The phenomenon is so pervasive that it has been proposed to use it in broadcasting our presence and communicating with other civilizations in the cosmos.[4] So, one could take either modality as primitive, and some people have strong preferences.[5]

Here is a formalization of a philosophical claim with both modalities:

Example (Deep philosophy?)**.** A student in Amsterdam in the 1970s once challenged me to formalize his profound conviction that "nothing is absolutely relative". Here is what I came up with:

$$\neg\Box(\Diamond\varphi \wedge \Diamond\neg\varphi)$$

which is equivalent, using some negation-pushing, to the implication $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$. This principle has been studied in modal logic as the "McKinsey Axiom", and we will see later what it means. Does this capture the student's wisdom? I leave that for you to judge.

Patterns should highlight important issues at a glance. Our discussion of duality was a good example. Here are some further useful distinctions that can be made with modal formulas:

Illustration (Important modal patterns)**.**

1. *Distinguishing scopes of modal operators.* The assertion

   "If you do p, you must also do q"

   has two non-equivalent readings, expressed by the two formulas

   $p \rightarrow \Box q$ ("narrow scope" for $\Box$ over q)
   $\Box(p \rightarrow q)$ ("wide scope" for $\Box$ over $\rightarrow$)

   This is a very popular confusion, from the Middle Ages right until the current literature. The two scopes come together in a principle

[4] Cf. the highly original monograph Freudenthal (1960) – written by a famous Dutch mathematician.

[5] I myself tend to be a box-man in axiomatic proof settings, and a diamond-man in semantic settings, depending on what works best.

that is called *Modal Distribution*:

$$\Box(p \to q) \to (\Box p \to \Box q)$$

whose validity, and controversial epistemic interpretation, will occupy us in later chapters.

Less confusing, but equally important, is the following modality-quantifier interchange:

"I know that someone appreciates me"

It has two possible readings, expressible as follows (with $\Box$ now standing for knowledge):

$\Box\exists x A(x,m)$	"*de dicto*" reading: modal qualification of a proposition
$\exists x \Box A(x,m)$	"*de re*" reading: assigning modal properties to objects

The two are not equivalent: though in this particular example, you might agree that the de re version implies the de dicto one. But I might know that I have at least one fan, even though, sadly, I will never find out in my entire life who that fan was.

2. *Iterations of modal operators.* Modal operators in natural language do not often occur stacked, except for American date lines like "I thought that we could, like, maybe, go out for a drink, sort of". It may be important for you to reduce this to its true logical form! More serious stacked versions occur with the epistemic interpretation of modality, where the following principles of "positive" and "negative" introspection have been hotly debated:

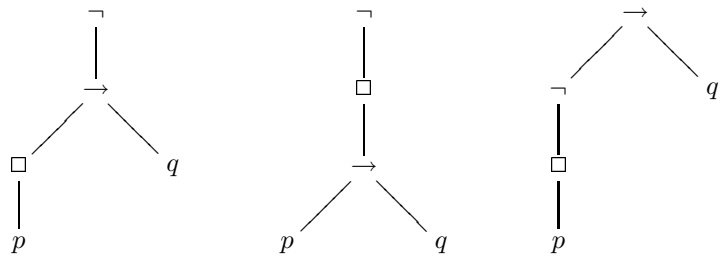
If I know, do I know that I know?	$\Box\varphi \to \Box\Box\varphi$
And if I don't know?	$\neg\Box\varphi \to \Box\neg\Box\varphi$

More generally, a logical formalism removes ambiguity from natural language, which has no brackets to disambiguate expressions, and no perspicuous recursion of operators.

Example (Disambiguation)**.** The un-bracketed flat symbol string $\neg\Box p \to q$ has three different modal readings:

$$\neg(\Box p \to q) \qquad \neg\Box(p \to q) \qquad (\neg\Box p \to q)$$

One can picture these readings more concretely as different syntax trees:



Logical syntax comes with some useful notions that measure expressive complexity. In particular, it is not the total number of modal operators that counts, but scoping:

Definition 2.1.2 (Modal depth)**.** The *modal depth* $md(\varphi)$ of a formula φ is the maximal length of a nested sequence of modal operators. This can be defined by the following recursion on our syntax rules:

1. $md(p) = 0$, 2. $md(\neg\varphi) = md(\varphi)$,
3. $md(\varphi \wedge \psi) = md(\varphi \vee \psi) = md(\varphi \rightarrow \psi) = \max(md(\varphi), md(\psi))$,
4. $md(\Diamond\varphi) = md(\Box\varphi) = md(\varphi) + 1$.

For instance, this works out to $md(\neg(\Box p \wedge \Diamond((p \wedge q) \wedge \neg\Diamond r))) = 2$.

As with logical systems in general, the mapping between intensional expressions in natural language and the formulas of modal logic is not unique – but we will not pursue this topic here. In any case, it is good to realize that modal expressions occur seldom as sentence operators – except in pompous phrases like "it is necessary that" – but rather in other linguistic categories such as adverbs ("maybe"), verbs ("can"), and adjectives ("possible"). Finally, the patterns discussed here serve two main purposes. Just like natural language, any logical language has several main uses, and at least two are crucial in logical theory: *describing situations, and formulating inferences*. The former perspective will soon take us to issues of modal definability and expressive power, the latter to axiomatic principles that can be employed in modal reasoning, a topic which we have not much emphasized yet.

Principles of modal reasoning are easily formulated in this notation. Most interpretations agree on basic modal laws like

$$\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi \qquad \text{or} \qquad \Diamond(\varphi \vee \psi) \leftrightarrow \Diamond\varphi \vee \Diamond\psi$$

These resemble laws for quantifiers – on the analogy

$$\Diamond : \exists \qquad\qquad \Box : \forall$$

But some modal laws change colours across different interpretations of $\Box$. The simple principle

$$\Box\varphi \to \varphi \qquad \text{("Veridicality")}$$

is valid for knowledge, but it is invalid for modalities expressing an obligation (what ought to be the case need not be true), or the result of an action (what will be made the case need not be true now). A modal formalism allows us to play on with this. For instance, the iterated principle $\Box(\Box\varphi \to \varphi)$ might well be considered valid on the obligation interpretation – though, true to form, it will not imply $\Box\varphi \to \varphi$.

2.2 Semantics of modal propositional logic

Our language will be interpreted over simple graph-like structures:

Definition 2.2.1 (Possible worlds models)**.** [6] A *possible worlds model* is a triple $\boldsymbol{M} = (W, R, V)$ of a non-empty set of *possible worlds* W, a binary *accessibility relation* R between worlds, and a *valuation map* V assigning truth values $V(p, s)$ in $\{0, 1\}$ to proposition letters p at worlds s.[7] Often, one works with *pointed models* $(\boldsymbol{M}, s)$ having one distinguished "current world" s serving as the "vantage point".

There are many interpretations for "possible worlds" s, ranging from metaphysical worlds[8] to worlds in science-fiction, states of a computer, board positions in chess, or deals in a card game. With such interpretations, the accessibility relation R can be universal (every world is accessible to every other), or constrained to game states reachable by later play, epistemic states constrained by what agents can see, points in space with their neighbours, etcetera. A particularly helpful geometrical interpretation is in machine diagrams or *process graphs*, where modal formulas describe possible evolutions starting from the current state of the process, with accessibility arrows for possible transitions.

Remark (Poly-modal languages)**.** Transitions in process graphs are often *labeled* with action types or events, so that we can have different accessibility relations R_a, and corresponding *labeled modalities* $\langle a\rangle, [a], \langle b\rangle, [b]$, etc. We will be using the latter "poly-modal languages" throughout these notes, but for now, we just continue with the unlabeled case, which is a pilot for all the rest.

Next, we make precise sense of the earlier-mentioned analogy between modal operators and quantifiers:

[6] By now, this is mainly a nostalgic name – but we stick to it like to any academic ritual. We will also use other common terms for these structures occasionally, like "relational models", or "Kripke models".

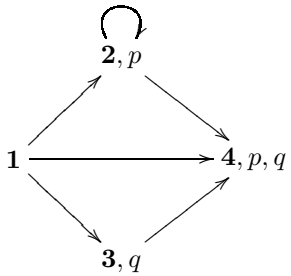
[7] Sometimes, we also write "$s \in V(p)$" for "$V(p, s) = 1$". This reflects a widely used identification of propositions with sets of worlds.

[8] Ridiculed in Voltaire's novel *Candide*, where survivors of the Lisbon Earthquake of 1755 are told by philosophers that they live in "the best of all possible worlds".

Definition 2.2.2 (Truth definition for the modal language)**.** A modal formula φ is *true* at world s in model $\boldsymbol{M} = (W, R, V)$ written $\boldsymbol{M}, s \models \varphi$, in virtue of this inductive clauses, following their syntax:[9]

$\boldsymbol{M}, s \models p$	iff	$V(p, s) = 1$
$\boldsymbol{M}, s \models \neg\varphi$	iff	*not* $\boldsymbol{M}, s \models \varphi$
$\boldsymbol{M}, s \models \varphi \wedge \psi$	iff	$\boldsymbol{M}, s \models \varphi$ *and* $\boldsymbol{M}, s \models \psi$
$\boldsymbol{M}, s \models \Box\varphi$	iff	*for all* t *with Rst*: $\boldsymbol{M}, t \models \varphi$
$\boldsymbol{M}, s \models \Diamond\varphi$	iff	*for some* t *with Rst*: $\boldsymbol{M}, t \models \varphi$

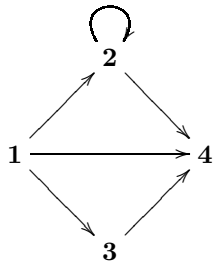
Example (Computing truth values of formulas in a model)**.** In the following process graph, some states have a unique outgoing arrow (the process is "deterministic" there), in others, there are several (the process is then "non-deterministic"), while there is also a "dead-lock" state without any outgoing arrows at all:



The valuation is written into the diagram in an obvious manner, by marking worlds. Here are some facts – for later reference, please note your own informal "model-checking procedure" here:

$\Diamond\Box p$	is true precisely in worlds	**1**, **2**, **3**
$\Box(q \to p)$	is true precisely in worlds	**2**, **3**, **4**

Next, strip this model of its valuation for p, q:

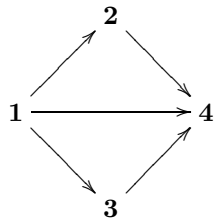


All four worlds can still be defined uniquely then by modal formulas true only at them:

[9] We only display clauses for two Boolean operators, but the remaining ones for disjunction and implication are similar.

4 is the only world satisfying	$\Box\bot$	("end point", "dead-lock")
3 is the only world with	$\Diamond\Box\bot \wedge \Box\Box\bot$	
1 is the only world with	$\Diamond(\Diamond\Box\bot \wedge \Box\Box\bot)$	
2 is the only world with	$\Diamond\Diamond\top \wedge \neg\Diamond(\Diamond\Box\bot \wedge \Box\Box\bot)$ [10]	

If we drop the loop, however, states 2 and 3 become modally indistinguishable – as you might try to prove informally:



Truth definitions are often considered somewhat boring, with clauses that seem trivial ("not" means *not*, etcetera). But there are interesting features to the process of evaluation sketched here. For a start, systematically, for any given modal formula, you can tabulate for all sub-formulas in which worlds they are true, working upwards in the construction tree of the formula. Thus, in the first model above:

p	is true precisely in worlds	**2**, **4**
$\Box p$	is true precisely in worlds	**2**, **3**, **4**
$\Diamond\Box p$	is true precisely in worlds	**1**, **2**, **3**

This method may be turned into an *efficient algorithm* for model-checking given modal formulas in given finite models, and we will compare this with the more complex case of first-order logic in a later chapter. We will give one more dynamic take on evaluation soon, but for now, we return to the reasoning side of modal logic.

2.3 Valid modal reasoning, a sneak preview

Universally valid principles are defined as usual:

Definition 2.3.1 (Modal validity)**.** A modal formula φ is *valid*, written as $\models \varphi$, if $\boldsymbol{M}, s \models \varphi$ for all models and worlds.

This is the sense in which the earlier two duality principles $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$, $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ were valid, or the mentioned modal distribution laws $\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$.

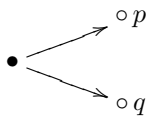
[10] There is often more than one natural choice for the world-defining formulas. For instance, we can define point **2** uniquely without cleverness, using the conjunction of the negated definitions for all the other points. More generally, in finite models, "the last world always comes for free".

One thing leads to another. As in propositional and predicate logic, there are strong dualities between the two modalities and disjunction/conjunction, resulting in automatic further laws for $\Diamond, \Box, \wedge, \vee$. Thus, switching operators, the obvious valid counterpart to $\Diamond(\varphi \vee \psi) \leftrightarrow \Diamond\varphi \vee \Diamond\psi$ is the principle $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$. On the same pattern of standing and falling together, invalid principles are:

$$\Diamond(\varphi \wedge \psi) \leftrightarrow \Diamond\varphi \wedge \Diamond\psi \qquad\qquad \Box(\varphi \vee \psi) \leftrightarrow \Box\varphi \vee \Box\psi$$ [11]

Invalidity of a formula is demonstrated concretely by displaying *counter-examples*, i.e., concrete models $\boldsymbol{M}$ with a world s where the invalid formula is not true:

Example (Counter-example to "$\Diamond$ over $\wedge$")**.** In the black dot in the following model $\boldsymbol{M}$, $\Diamond p$ and $\Diamond q$ are true, but $\Diamond(p \wedge q)$ is not:



We will talk later about algorithmic methods for finding such counter-examples to invalid principles more systematically, turning an "art" of finding counter-examples into a "science".

Additional validities arise as axioms for modal logics with special interpretations, in the form of special constraints on the accessibility relation R. For instance, imposing the earlier Veridicality as a valid principle in our models (in Chapter 9, we will make it precise what this means) has the following effect:

$\Box\varphi \to \varphi$ says that the relation R is *reflexive*: $\forall x Rxx$

Another well-know example makes accessibility work in one step:

$\Diamond\Diamond\varphi \to \Diamond\varphi$ says R is *transitive*: $\forall x \forall y (Rxy \to \forall z (Ryz \to Rxz))$

Observations like these lead to the famous "landscape of modal logics", with axioms of different strengths on top of the universal validities, and ubiquitous names like "T", "$S4$", "$S5$", whose botanical exploration has been a hallmark of modal logic in its earlier phases. We, too, will study possible modal axioms in more detail in Chapters 8 and 9.

[11] Actually, in both cases, one implication is valid: which one? This is the so-called *monotonicity* of $\Diamond$ and $\Box$.

2.4 Semantics as dynamic procedure: modal evaluation games

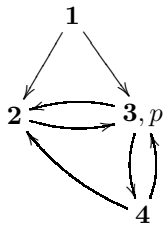
But there is still more to our "simple" truth definition! Evaluating a modal formula φ may be viewed as a dynamic procedure, which is of interest by itself. One particularly lively way of doing this is by means of a *game*, a technique of growing importance in logic today.

Definition 2.4.1 (Modal evaluation game)**.** Let $\boldsymbol{M}, s$ be a model, and φ a modal formula. We define a two-person game $game(\boldsymbol{M}, s, \varphi)$ between a player *Verifier* (V), claiming that φ is true, and *Falsifier* (F) claiming φ is false. The game starts at the world s. Each move is dictated by the main operator of the formula φ and we move to its sub-formulas: the total length is bounded by the operator depth:

atom p	*test* p at s: if true, then V wins, if false, then F wins,
$\varphi \vee \psi$	V *chooses* a disjunct and play continues with that,
$\varphi \wedge \psi$	F *chooses* a conjunct and play continues with that,
$\Diamond\varphi$	V *picks* an R-*successor* t of the current world, play continues with φ at t,
$\Box\varphi$	F *picks* an R-*successor* t of the current world, play continues with φ at t,
$\neg\varphi$	players switch roles in the φ game.

A player also loses when (s)he must pick a successor, but cannot do so.

Example (A complete game tree)**.** Here is the complete game tree for the modal formula $\Box(\Diamond p \vee \Box\Diamond p)$ played starting at state **1** in the following modal model (you may want to check beforehand if it is true):



In Figure 1, we draw the game nodes, plus the player that is to move next, plus the relevant formula. Bottom leaves have "win" if Verifier wins (the atom is true there), "lose" otherwise.

In this game, both players have winning runs. But even though Verifier has fewer winning runs than Falsifier, in the response dynamics of this game tree, the advantage is still clearly for her. The important game-theoretic concept here is that of a *strategy*, a rule which tells a player what to do at each of her turns. For, Verifier has a "*winning*

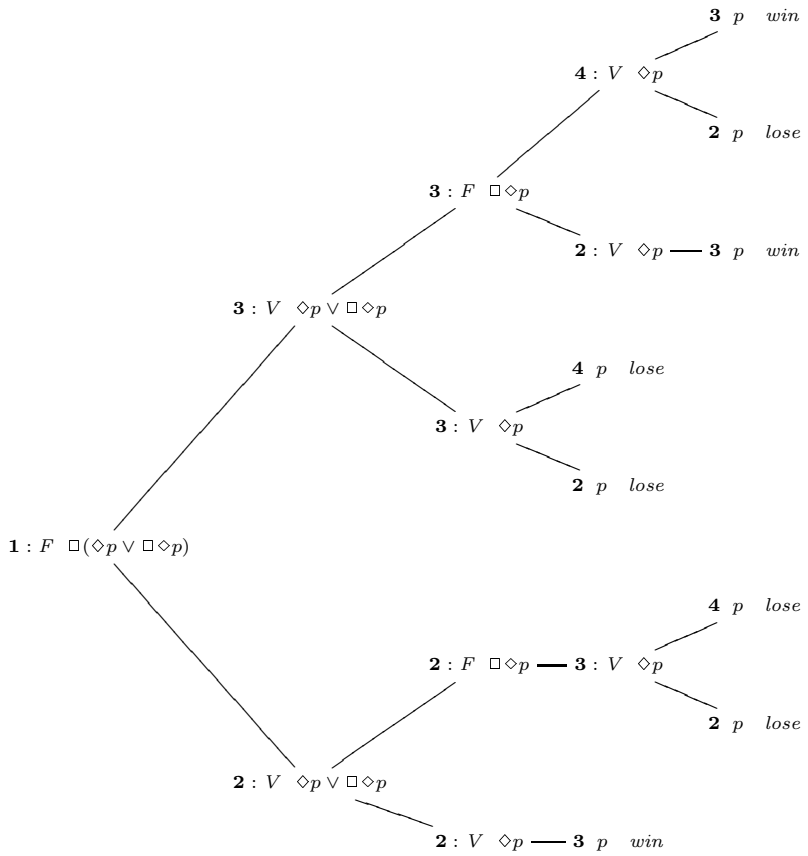


FIGURE 1 A complete game tree for the example on page 19.

strategy" in this game, a rule for playing that ensures a win for her whatever Falsifier does. The moves prescribed by this strategy are indicated by the double lines in the following picture:

Example (A winning strategy in a modal game tree)**.** Figure 2 shows how Verifier should responde to Falsifier's moves.

Note that a strategy can encode subtle interactive behaviour. For instance, Verifier has to hand the initiative to Falsifier at state 3 on the right in the game tree if she is to win! Moreover, the full picture of interaction in this game would look at all possible strategies; for instance, note that Verifier also has a *losing strategy*, ensuring that she loses whatever Falsifier does.

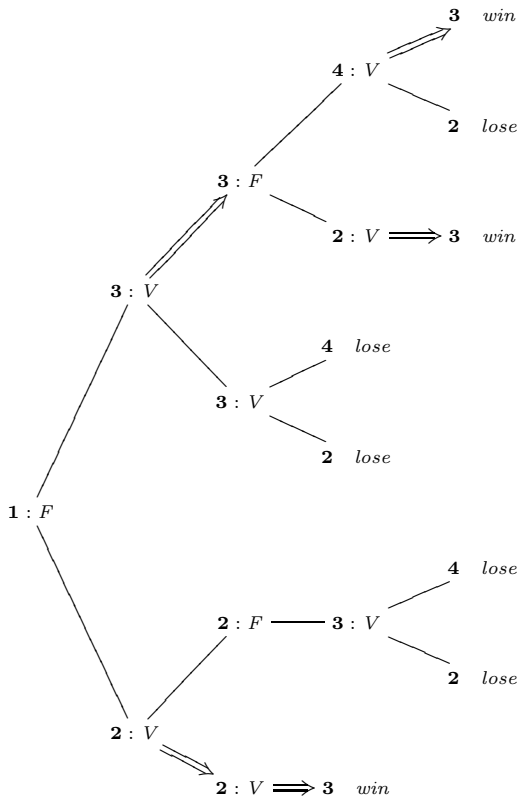


FIGURE 2 Winning strategy for Verifier in the example on page 20.

The remaining operation of negation as role switch involves the same tree structures. For instance, the game for the formula $\neg\Box(\Diamond p \vee \Box\Diamond p)$ looks exactly like the game tree that we drew for $\Box(\Diamond p \vee \Box\Diamond p)$, but with (a) all V, F turn markings interchanged at intermediate nodes, and (b) all "win", "lose" markings switched at end nodes.

Note also that these game trees are *themselves* modal models as we defined them here, with nodes as worlds, and moves as accessibility relations. This gives a foothold for applications of modal logic to *game theory*. We pursue this in Chapter 17 of our lectures. For the moment, games are just a didactic tool in our presentation of modal logic.[12]

[12]In fact, evaluation games in the style explained here work just as well for first-order logic, and many other logical systems.

Finally, the reason for Verifier's advantage in the above game is intuitively clear: after all, she is defending a *true statement* in the model $\boldsymbol{M}$ at the initial state s. As a simple challenge to the reader, we state the general fact behind this observation:

Key Lemma. The following two assertions are equivalent for all modal formulas and pointed models:

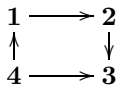
1. φ is true in $\boldsymbol{M}$ at s,
2. V has a winning strategy in $game(\boldsymbol{M}, s, \varphi)$.

Proof. The proof is a simple induction on the syntactic construction of the formula φ and going through its successive cases will make you appreciate the very strong connection between the logical operators in our basic modal language and game moves. □

If the formula is false, then Falsifier has a winning strategy, and hence modal evaluation games are "*determined*": one of the two players has a winning strategy. We will prove in Chapter 17 why determinacy must always hold for logical games of the present kind.

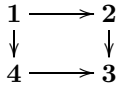
Exercises Chapter 2

1. (a) For each point s in the following model, give a modal formula that is only true at s:

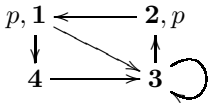


   Use modal formulas involving only "true" ($\top$) and "false" ($\bot$).

   (b) Try the same in the following model. Explain your difficulty.



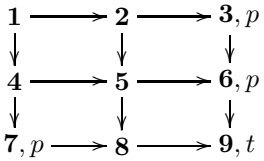
2. (a) Determine in which states of the following model the modal formula $\Diamond\Box\Diamond p$ is true:



   (b) Give a complete game tree for the evaluation game for $\Diamond\Box\Diamond p$ in this model starting in state **1**.

   (c) Indicate the winning strategies for the player who has them.

3. ("Treasure Island") Consider the following model with 9 states, and an accessibility relation allowing one step east or south (insofar as possible) from each point. State **9** has a treasure, marked by the proposition letter t, pirates are standing at **3**, **6**, and **7**, marked by the proposition letter p:



   (a) In which states are the following modal formulas true?

$$\Diamond t, \quad \Diamond\Box t, \quad \Diamond p, \quad \Box\Diamond p$$

   (b) For each state, find a modal formula true only there.