Duality Theory in Logic

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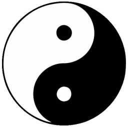
Cool Logic
14th December

Duality in General



"Duality underlines the world"

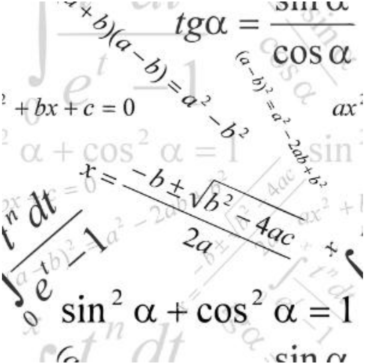
- *"Most human things go in pairs"* (Alcmaeon, 450 BC)

  Existence of an entity in seemingly different forms, which are strongly related.
- *Dualism* forms a part of the philosophy of eastern religions.
- In Physics : Wave-particle duality, electro-magnetic duality, Quantum Physics,...

Duality in Mathematics

- Back and forth mappings between dual classes of mathematical objects.
  - Lattices are self-dual objects
  - Projective Geometry
  - Vector Spaces
- In logic, dualities have been used for relating syntactic and semantic approaches.

Algebras and Spaces



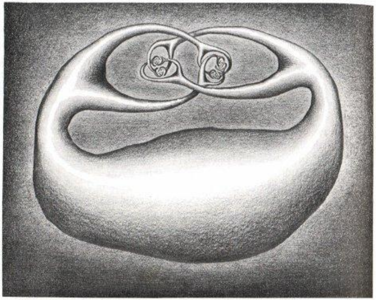
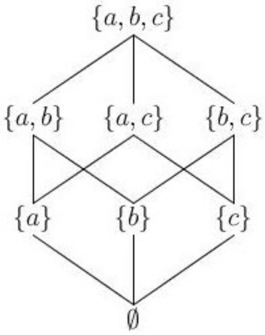


FIG. 4–11. The Alexander horned sphere.

Logic fits very well in between.

Algebras

- Equational classes having a domain and operations .
- Eg. Groups, Lattices, Boolean Algebras, Heyting Algebras . . .,
- Homomorphism, Subalgebras, Direct Products, Variety, . . .

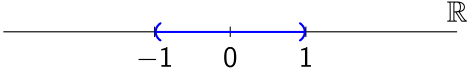


Power Set Lattice (BDL)

- BA = (BDL + ¬) s.t. $a \vee \neg a = 1$

(Topological) Spaces

- Topology is the study of spaces.
  A toplogy on a set X is a collection of subsets (open sets) of X, closed under arbitrary union and finite intersection.

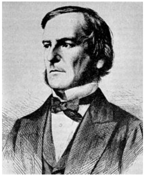


  Open Sets correspond to neighbourhoods of points in space.
- Metric topology, Product topology, Discrete topology ...
- Continuous maps, Homeomorphisms, Connectedness, Compactness, Hausdorffness,...

"I dont consider this algebra, but this doesn't mean that algebraists cant do it." (Birkhoff)

A brief history of Propositional Logic

- Boole's *The Laws of Thought* (1854) introduced an algebraic system for propositional reasoning.
- Boolean algebras are algebraic models for Classical Propositional logic.
- Propositional logic formulas correspond to *terms* of a BA.



George Boole (1815-1864)

$$\vdash_{CPL} \varphi \quad \Leftrightarrow \quad \models_{CPL} \varphi \quad \Leftrightarrow \quad \models_{BA} \varphi = \top$$

Representation in Finite case

- Representation Theorems every element of the class of structures X is isomorphic to some element of the proper subclass Y of X
    - Important and Useful (algebraic analogue of completeness)
    - Cayley's theorem, Riesz's theorem, ...

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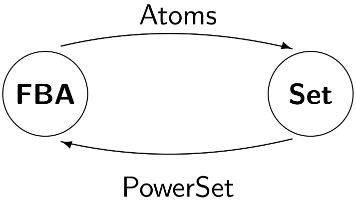
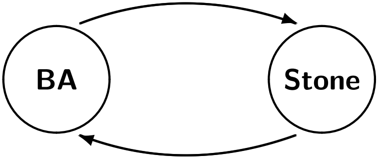


- Map every element of the algebra to the set of atoms below it $f(b) = \{a \in At(B) \mid a \leq b\}$ for all $b \in B$
  $\mathbf{FBA} \cong \mathcal{P}(\mathbf{Atoms})$

Stone Duality

'A cardinal principle of modern mathematical research maybe be stated as a maxim: "One must always topologize" '.

-Marshall H. Stone



$$\mathbf{BA} \cong \mathbf{Stone}^{\mathrm{Op}}$$

Stone's Representation Theorem (1936)

M. H. Stone (1903-1989)

From Spaces to Algebras and back

- Stone spaces Compact, Hausdorff, Totally disconnected
  eg. $\mathbf{2}^A$, Cantor set, $\mathbb{Q} \cap [0, 1]$
- Stone space to BA
  Lattice of clopen sets of a Stone space form a Boolean algebra

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- Stone space to BA
  Lattice of clopen sets of a Stone space form a Boolean algebra
- BA to Stone space
  Key Ideas :
  (i) Boolean algebra can be seen a Boolean ring (Idempotent)
  (ii) Introducing a topology on the space of *ultrafilters* of the Boolean ring

What is an ultrafilter ?

From Spaces to Algebras and back

- A *filter* on a BA is a subset F of BA such that
  - $1 \in F, 0 \notin F$;
  - if $u \in F$ and $v \in F$, then $u \wedge v \in F$;
  - if $u, v \in B$, $u \in F$ and $u \leq v$, then $v \in F$.

  In short, its an upset, closed under meets.
  An *ultrafilter* U, is a filter such that either $a \in U$ or $\neg a \in U$.
- Example: Let $\mathcal{P}(X)$ be a powerset algebra Then the subset $\uparrow \{x\} = \{A \in \mathcal{P}(X) \mid x \in A\}$ is an ultrafilter.
  Non-principal ultrafilters exist (Axiom of Choice)

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- (i) Map an element of B to the set of ultrafilters containing it $f(b) = \{u \in S(B) \mid a \in u\}$
  (ii) Topology on $S(B)$, is generated by the following *basis* $\{u \in S(B) \mid b \in u\}$ where $b \in B$

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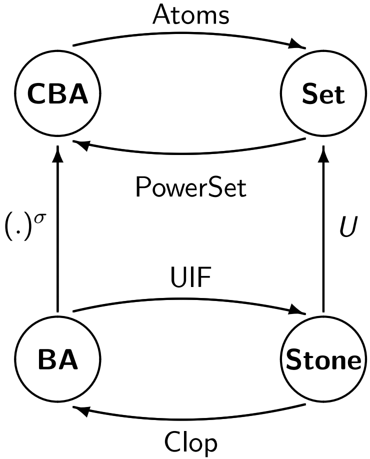
- Morphisms and Opposite (contravariant) Duality

The Complete Duality

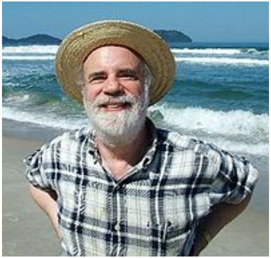


- Canonical extensions
- Stone-Céch Compactification

Modal logic as we know it

Kripke had been introduced to Beth by Haskell B. Curry, who wrote the following to Beth in 1957

"I have recently been in communication with a young man in Omaha Nebraska, named Saul Kripke... This young man is a mere boy of 16 years; yet he has read and mastered my Notre Dame Lectures and writes me letters which would do credit to many a professional logician. I have suggested to him that he write you for preprints of your papers which I have already mentioned. These of course will be very difficult for him, but he appears to be a person of extraordinary brilliance, and I have no doubt something will come of it."



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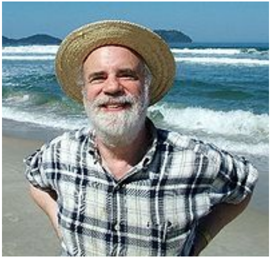


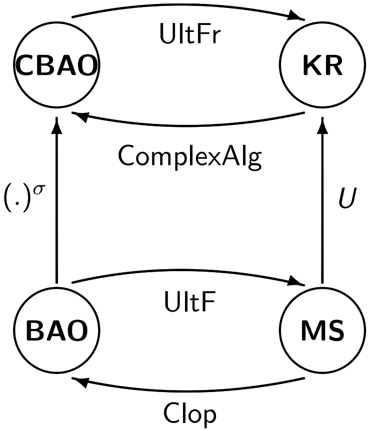
Saul Kripke

Saul Kripke, *A Completeness Theorem in Modal Logic*. J. Symb. Log. 24(1): 1-14 (1959)

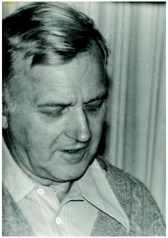
Modal logic before the Kripke Era

- C.I. Lewis, *Survey of Symbolic Logic*, 1918
  (Axiomatic system **S1-S5**)
  - *Syntactic era* (1918-59)
    Algebraic semantics, JT Duality, ...
  - *The Classical era* (1959-72)
    "Revolutionary" Kripke semantics, Frame completeness,...
  - *Modern era* (1972- present)
    Incompleteness results (FT '72, JvB '73), Universal algebras in ML, CS applications,...
- Modal Algebra (MA) = Boolean Algebra + Unary operator $\Diamond$
  1. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
  2. $\Diamond\bot = \bot$
  3. $\Diamond(a \rightarrow b) \rightarrow (\Diamond a \rightarrow \Diamond b)$ (Monotonicity of $\Diamond$)

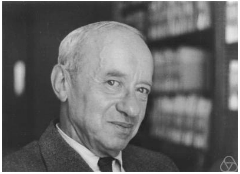
Jóhsson-Tarski Duality



Jóhnsson-Tarski Duality (1951-52)



Bjarni Jóhnsson



Alfred Tarski (1901-1983)

Modal Spaces and Kripke Frames

- Key Idea: We already know, ultrafilter frame of the BA forms a Stone space. For BAO, we add the following relation between ultrafilters

$$Ruv \text{ iff } fa \in u \text{ for all } a \in v$$

- Descriptive General Frames
  - Unify relational and algebraic semantics
  - DGF = KFr + *admissible* or *clopen* valuations
  - Validity on DGF $\Rightarrow$ Validity on KFr
    Converse (Persistence) only true for Sahlqvist formulas

Algebraic Soundness and Completeness

- Theorem: Let Σ set of modal formulas. Define

$$V_\Sigma = \{\mathbb{A} \in BAO \mid \forall\varphi(\varphi \in \sigma \Rightarrow \mathbb{A} \models \varphi = \top)\}$$

Then for every ψ, $\vdash_K \psi$ iff $V_\Sigma \models \psi = \top$.

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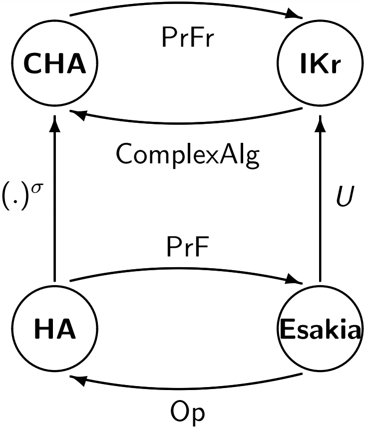
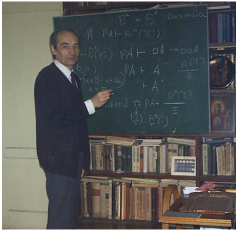
- Canonical Models are Ultrafilter frames of Lindenbaum-Tarski algebra
- Disjoint Union $\leftrightarrow$ Product
  Bounded Morphic image $\leftrightarrow$ Subalgebras
  Generated subframe $\leftrightarrow$ Homomorphic image

Esakia Duality



Esakia Duality (1974)



Leo Esakia
(1934-2010)

- Useful in characterizing *Intermidiate logics*.

List of Dual Structures in Logic

Duality	**Algebra**	**Space**	**Logic**
Priestly Duality	**DL**	**Priestly**	negation free CL
Esakia Duality	**HA**	**Esakia**	IPL
Stone Dualtiy	**BA**	**Stone**	CPL
Jóhnsson-Tarski Duality	**MA**	**MS**	ML
...			

Frame Definability and Correspondence

- Elementary class of frames
- Reflexive, Transitive, Antisymmetric, . . .

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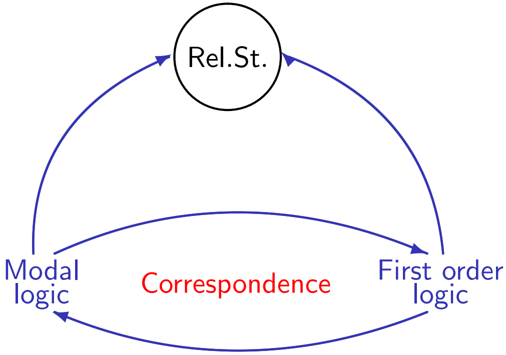
  - It's an undecidable problem [Chagrova '89]
  - Sahlqvist formulas provide *sufficient* conditions

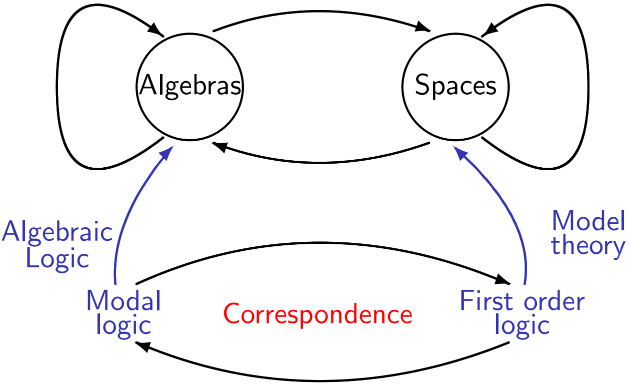
Correspondence Theory

- Johan's PhD thesis *Modal Correspondence Theory* in 1976.
- **Correspondence**
  Provides *sufficient* syntactic conditions for first order frame correspondence eg. Sahlqvist formulas.
- **Completeness**
  Sahlqvist formulas are *canonical* and hence axiomatization by Sahlqvist axioms is complete.

Classical Correspondence



Correspondence via Duality

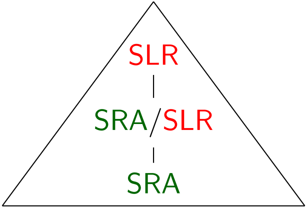


Correspondence via Duality

Key Ideas

(i) Use the properties of the algebra to drive the correspondence mechanism.
(ii) Use the (order theoretic) properties of the operators to define sahlqvist formulas

Eg. $\Box p \to p$
iif $\Box p \leq p$
iff $p \leq (\Box)^{-1} p$ ($\Box$ as SRA)
iff $\forall \mathbf{i} \forall \mathbf{j}\ \ \mathbf{i} \leq p\ \&\ (\Box)^{-1} p \leq \mathbf{m}$
iff $\forall \mathbf{i} \forall \mathbf{j}\ \ (\Box)^{-1} \mathbf{i} \leq \mathbf{j}$
iff $\forall x,\ x \in R[x]$



Sahlqvist formula

Advantages

(i) Counter-intuitive frame conditons can be easily obtained (eg. Löb's axiom)
(ii) The approach generalizes to a wide variety of logics.

Point Free Topology

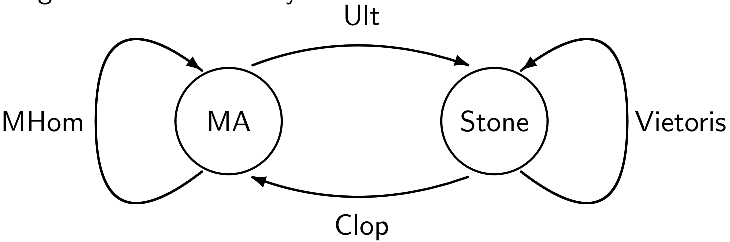
- Point free Topology
  - Open sets are first class citizens
  - Lattice theoretic (algebraic) approach to topology
  - Sober spaces and Spatial locales
- Gelfand dualtiy
  - locally KHaus and the C*-algebra of continuous complex-valued functions on X
  - Understanding spaces by maps.
- Algebraic Topology?

Peter T. Johnstonne

Israel Gelfand (1913-2009)

Coalgebras

- *Modal logics are Colagebraic* [CKPSV '08]
  Kripke frames as transition systems.
- Coalgebraic Stone Duality

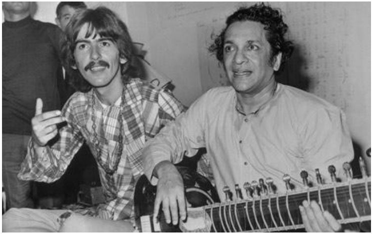


$$\mathbf{MA} \cong \mathsf{Alg}(\mathsf{Clop} \circ \mathcal{V} \circ \mathsf{Ult}) \cong \mathsf{CoAlg}(\mathcal{V})^{op}$$

References

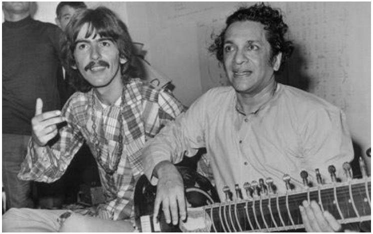
- *Lattices and Order*, B. Davey and H. Priestly
- *Modal Logic*, [BRV] Ch. 5
- *Stone Spaces*, P. T. Johnstonne
- Mathematical Structures in Logic

The Beatles



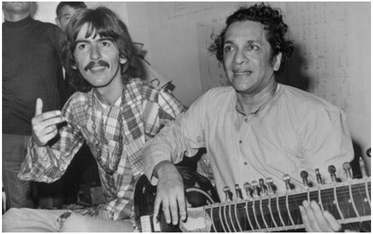
"Within You Without You" is a song written by George Harrison, released on The Beatles' 1967 album, Sgt. Pepper's Lonely Hearts Club Band.

The Beatles



"...And the time will come when you see
We're all one, and life flows on within you and without you"

The Beatles



“. . . And the time will come when you see
We’re all one, and logic flows on within you and without you”
(summarizes duality theory quite well !)