Horrors with and without the Axiom of Choice

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Overview

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Definition

The *Axiom of Choice* is the following statement: Let I be a set. Let $(X_i)_{i \in I}$ be an I-indexed family of non-empty sets. Then there is an I-indexed family $(x_i)_{i \in I}$ of elements such that for every $i \in I$, $x_i \in X_i$.

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An equivalent: A and B are sets, and $f : A \to B$ a surjection. Then there is an injection $g : B \to A$ such that for every $a \in A$, $g(f(a)) = a$.

Con(**ZF**) → Con(**ZFC**) (Gödel).

Con(**ZF**) → Con(**ZF**¬**C**) (Cohen).

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Let B^3 be the unit ball in $\mathbb{R}^3$. Then it can be decomposed into finitely many pieces and these pieces rearranged in $\mathbb{R}^3$ by only using rotations and translations into two disjoint copies of B^3.

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But if you think that things are just dandy without **AC**, wait till we get to the end of the presentation.

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Aside: **CH** is the question “Does $\mathbb{R}$ have size $\aleph_1$?”

Basic properties of a measure

Definition

A *measure* on $\mathbb{R}$ is a (possibly partial) function $\mu : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ satisfying:

(i) The lengths of intervals should be intuitive: $\mu([0,1]) = 1$

(ii) Singletons should be insignificant: $\mu(\{*\}) = 0$

(iii) Countable additivity: Let $(S_n)_{n\in\mathbb{N}}$ be disjoint subsets of $\mathbb{R}$. Then $\mu(\biguplus_{n\in\mathbb{N}} S_n) = \Sigma_{n\in\mathbb{N}}\mu(S_n)$.

(iv) Translation Invariance: Let $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$. Let $T = \{s + r; s \in S\}$. Then $\mu(S) = \mu(T)$.

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Observations:

(i) Each equivalence class has the same size as $\mathbb{Q}$ i.e. $\aleph_0$.

(ii) So there are uncountably many equivalence classes.

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(v) Therefore, $1 \leq \Sigma_{q\in\mathbb{I}}\mu(V_q) \leq 3$.

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(vi) So $\mu(V) \geq 0$, but $\Sigma_{q\in\mathbb{I}}\mu(V_q) = \Sigma_{q\in\mathbb{I}}\mu(V) \leq 3$. Contradiction.

Groups

Definition

A *group* is a set G together with a binary operation $\bullet$ which satisfies the following laws:

(i) Closure: If $a, b \in G$ then $a \bullet b \in G$.

(ii) Identity: There is an element $e \in G$ such that for all $g \in G$, $e \bullet g = g$.

(iii) Associativity: If $a, b, c \in G$, then $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.

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Example: $(\mathbb{Z}, +, 0)$. Given a set S, the set of its permutations. SO_3, the set of rotations of $\mathbb{R}^3$. $\mathbf{F}_2 \subseteq SO_3$.

Group actions

Definition

Let G be a group and X a set. A *(left) group action* of G on X is a binary operator $\circ : G \times X \to X$ satisfying the following laws:

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Example: Any group acting on itself. Given a set S, any subgroup of its permutation group, acting on S. SO_3, acting on $\mathbb{R}^3$.

Decomposing and Assembling

Let G act on X. Let E, F subsets of X. Say E and F are *equidecomposable* via G with m pieces (denoted $E \sim_G F$) if :

(i) There are $g_1, \cdots g_m$ in G and $A_1, \cdots A_m$ pairwise disjoint subsets of E such that:

(ii) $E = \biguplus_{i \leq m} A_i$ and $F = \biguplus_{i \leq m} g_i A_i$.

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Aside: *Amenable* groups.

The broken circle

Example: S^1 a proper circle and $S^1 \setminus \{*\}$ a broken circle, then $S^1 \setminus \{*\} \sim S^1$ via SO_2, the group of rotations of $\mathbb{R}^2$ with 2 pieces.

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   2π is irrational so this rotation *freely* generates an infinite set. That is, $e^{im} \neq e^{in}$ for $m \neq n$.

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  e^{-i} sends $e^{i(n+1)}$ to e^{in}.

$\mathbf{F}_2$

Definition

The free group on the 2 generators $\{a, b\}$ is defined as follows:

(i) The base set consists of all finite strings that can be formed from the alphabet $\{a, a^{-1}, b, b^{-1}\}$ which do not contain the substrings $aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b$.

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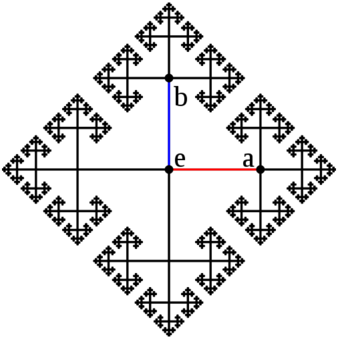
But this is *not* the same as $(\mathbb{Z} \times \mathbb{Z}, (+, +), (0, 0))$.

A picture of $\mathbf{F}_2$



Deconstructing $\mathbf{F}_2$, Reconstructing $\mathbf{F}_2$, $\mathbf{F}_2$

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(v) Easy to verify that $\mathbf{F}_2 = G_1 \uplus aG_2 = G_3 \uplus bG_4$.

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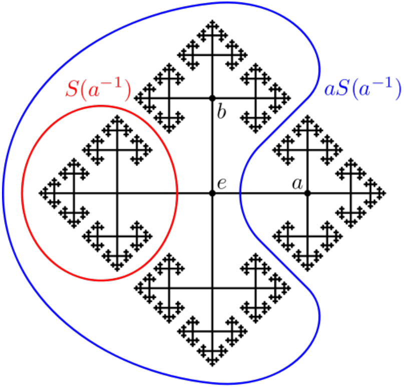

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(iii) The following is then an equivalence relation on $S^2 \setminus D$: $x \sim y$ iff there is a $f \in \mathbf{F}_2$ such that $f \circ x = y$.

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(i) Let $D \triangleq \{x \in S^2; \exists f \in \mathbf{F}_2(f \bullet x = x)\}$.

(ii) Let G_1, G_2, G_3, G_4 be as previously.

(iii) The following is then an equivalence relation on $S^2 \setminus D$: $x \sim y$ iff there is a $f \in \mathbf{F}_2$ such that $f \circ x = y$.

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(vii) Hence, $S^2 \setminus D = \Omega_1 \uplus a\Omega_2 = \Omega_3 \uplus b\Omega_4$.

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And the number of pieces we've needed is 8.

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But this easily follows from $S^1 \setminus \{*\} \sim S^1$.

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- Aside: The Downward Löwenheim-Skolem theorem is equivalent to **AC** (Tarski).

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(v) Aside: **AC** says that $\mathbb{R}$ has a definite size in this sense, but this size can be almost anything by Cohen.

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In fact, *all* partial orders can be embedded as cardinalities.

Let I be a set and for every $i \in I$, X_i a non-empty set. Then without **AC**, $\prod_{i\in I} X_i$ may be empty.

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(i) Without **AC**, there can be an infinite tree with no leaves, but no infinite paths either.

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(ii) Without **DC** (the same as the above statement about trees), it is not possible to develop a satisfactory theory of real analysis or measure theory.

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(i) Assume all subsets of $\mathbb{R}$ are measurable. Then $\omega_1 \not\leq 2^\omega$ by Raisonnier.

(ii) Then $\aleph_1 + 2^\omega > 2^\omega$. The injection from right to left is trivial. The reverse injection is not possible.

(iii) But there is a partion of 2^ω (actually $\mathcal{P}(\omega + \omega)$) into $\aleph_1 + 2^\omega$ many parts. Two well-ordering of ω are mapped to the same set if their ordertype is the same, non-wellorderings are mapped to singletons.

How I learned to stop worrying and...

1. $\mathbb{R}$ may be a countable union of countable sets (Feferman-Lévy). But not countable, by Cantor's theorem!
2. $\aleph_1$ can be a *large* cardinal (Jech).
3. $\aleph_1$ can be a countable union of countable sets (Lévy).
4. *Every* infinite set may be a countable union of smaller sets (Gitik).
5. There is a model of **ZF** in which there is no function C with the following properties: for all X and Y,
   (i) $C(X) = C(Y)$ if and only if $|X| = |Y|$
   (ii) $|C(X)| = |X|$

Be careful! :-)

There is richer structure without **AC**, somewhat similar to the greater structure in Intuitionistic Logic.

To extend the analogy further, weak forms of **AC** play the part of weak forms of **LEM**.
Aside: In a set theory with **AC**, you can show that **LEM** (or weaker forms) holds (Diaconescu/Goodman-Myhill).

But perhaps this is too much structure? And counterintuitive structure at times.

The Axiom of Choice is a subtle beast. Use it, but with care.