Are There True Contradictions?

Paraconsistent Logic and Dialetheism

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Outline of the talk

- Introduction
- The semantic paradoxes and their attempted solutions
- The Logic of Paradox
- What is so bad about contradictions?

A quote

"Indeed, even at this stage I predict a time when there will be mathematical investigations of calculi containing contradictions, and people will actually be proud of having emancipated themselves even from consistency."

—Ludwig Wittgenstein

Preliminary definitions I

Definition

Dialetheism is a philosophical position according to which there are true sentences of the form $\varphi \wedge \neg\varphi$. We call these sentences dialetheia, or true contradictions.

- Weak dialetheism holds that certain sentences are best explained by calling them true contradictions. This is also called semantic dialetheism.
- Strong dialetheism is the view that the world itself is somehow inconsistent.

Preliminary definitions II

Definition

A paradox is an argument which proceeds from premises which appear true via a number of steps which appear valid, to a conclusion which is nevertheless untrue.

- Dialetheism is a response to certain paradoxes.

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- These are called "semantic paradoxes" because they involve the notion of truth. Dialetheism is accepting the paradoxical argument.

What do we need to generate the Liar Paradox?

1. Self-reference, or something equivalent.
2. A truth predicate with Capture and Release: $T(\ulcorner \varphi \urcorner) \leftrightarrow \varphi$.
3. The law of excluded middle: $\varphi \vee \neg\varphi$.

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- Philosophers have tried to doubt all of these things in order to escape the paradox. Almost everyone accepts (2), however.

The Tarskian Solution

- We restrict self-reference by stipulating that a truth definition must be given in a stronger meta-language than the object language.
- This gives rise to a hierarchy of languages each with its own truth predicate.
- On this picture, the Liar sentence is not well-formed: "This$_0$ sentence is true$_1$".

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- Intutively it just seems false that a natural language has hierarchies.
- Yesterday Julia said: "Everything Ásgeir will say in his talk is false".
- Now I say: "Everything Julia said yesterday is true."
- Kripke's conclusion: "[It is] fruitless to look for an intrinsic criterion that will enable us to sieve out—as meaningless, or ill-formed—those sentence which lead to paradox."

Problems with the Tarkian Solution II

- The real problem is not that we can't avoid the paradoxes formally.
- A real solution should tell us which step in the argument is wrong, and why.
- This explanation should be independent of the Liar paraxdox, i.e. not just designed to avoid it.

Kripke's Solution I

- The main idea is that truth must be grounded in non-semantic facts: "It is true that snow is white" is true because snow is in fact white.
- The Liar sentence never refers to anything but language and is ungrounded.
- It therefore shouldn't have a truth value.

Kripke's Solution II

- Start with a classical model without a truth predicate.
- Build a hierarcy of languages, each extending the truth predicate.
- Take the smallest fixed point and evaluate truth there: The Liar doesn't have a truth value!
- The solution is not ad hoc, because we have an explanation why the Liar doesn't have a truth value: it is not grounded.

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- This is called a strengthened Liar, or Revenge.

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- Dialetheists reject this principle, and thus avoid trivialism.
- But we need a new logic which does not have this rule.

Semantics for the Logic of Paradox I

The Logic of Paradox is a three valued logic, but instead of truth value gaps, there are sentences than can be true and false. This gives rise to the following definition:

Definition

Let $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$, where $\mathcal{D}$ is the domain and $\mathcal{I}$ a total function from the atomic sentences to $\{0, 1, \frac{1}{2}\}$.

The truth values are interpreted to mean:

- 0 means 'false but not true'.
- 1 means 'true but not false'.
- $\frac{1}{2}$ means 'both true and false'. We also call these sentences 'paradoxical'.

Semantics for the Logic of Paradox II

We define the valuation function $\mathcal{V}$ in the following way:

Definition

- $\mathcal{V}_{\mathcal{M}}(\varphi) = \mathcal{I}(\varphi)$ if φ is atomic.
- $\mathcal{V}_{\mathcal{M}}(\neg\varphi) = 1 - \mathcal{V}_{\mathcal{M}}(\varphi)$.
- $\mathcal{V}_{\mathcal{M}}(\varphi \vee \psi) = max(\mathcal{V}_{\mathcal{M}}(\varphi), \mathcal{V}_{\mathcal{M}}(\psi))$
- $\mathcal{V}_{\mathcal{M}}(\varphi \wedge \psi) = min(\mathcal{V}_{\mathcal{M}}(\varphi), \mathcal{V}_{\mathcal{M}}(\psi))$

This just means that the negation of a paradoxical sentence is paradoxical, the conjunction of a false sentence and a paradox is false, etc.

Semantic consequence and logical truth

In order to avoid explosion, we need to redefine our consequence relation.

Definition

Let Δ be a set of sentences. Then $\Delta \vDash \psi$ iff $\forall \mathcal{M}$ and $\forall \varphi_i \in \Delta$: $\mathcal{V}_{\mathcal{M}}(\varphi_i) \geq \frac{1}{2}$ then $\mathcal{V}_{\mathcal{M}}(\psi) \geq \frac{1}{2}$

Consequence is not preservation of *truth and not falsity* but just *preservation of truth*.

Theorem

Every classical tautology is an LP-tautology.

Some deductions that do not hold

The following is *not* valid:

- Explosion: $\varphi \wedge \neg\varphi \vDash \psi$
- Transitivity: $\varphi \to \psi, \psi \to \chi \vDash \varphi \to \chi$
- Reductio ad absurdum: $\varphi \to (\psi \wedge \neg\psi) \vDash \neg\varphi$
- Modus Ponens: $\varphi, \varphi \to \psi \vDash \psi$

Now we must be in trouble. How is even anything resembling ordinary reasoning, much less mathematics, possible without these principles?

The valid/quasi-valid distinction

- These rules of inference are not *generally* valid, but are valid if the premises are classically true.
- These we call quasi-valid.
- Graham Priest suggests that we can just keep on using them anyway, with provisio.

The provisio

- "Unless we have specific grounds for believing that paradoxical sentences are occuring in our argument, we can allow ourselves to use both valid and quasi-valid inference."
- In practice this is not so different: You can never be more sure of your conclusion than you are of your premises.
- Now we just have to make sure our premises are true *and* not paradoxical.
- So we can "have our cake and eat it".

Summary so far

- It's easy to avoid trivialism.
- The Logic of Paradox can prove all classical tautologies.
- In non-paradoxical situations it is in fact the same as classical logic.
- But we have to accept that there are true contradictions.

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  - That only shows that our logic isn't classical.
- Surely contradictions cannot be true!
  - Why not?
- If contradictions were acceptable, people could never be criticized.
  - The dialetheist doesn't maintain that all contradictions can be true, only very special ones. Most will be false, and rationally acceptable as such.

Final remarks

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The idea that contradictions might be true shouldn't be surprising. After all, our language has all the necessary tools to generate inconsistency:

- A truth predicate.
- Self-reference.
- This isn't necessarily a metaphysical point, but a linguistic one: our language is inconsistent.
- And so what?

A joke

A Catholic priest, a Protestant priest and Graham Priest walk into a bar. The Catholic says to the bartender, "I'll have a whiskey". The Protestant thinks a little while and then says: "I don't think I shall". Graham Priest says, "I'll have what they're having."