Kripke models for first-order intuitionistic logic

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Overview

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The Developement of Intuitionism

- **1901:** B. Russell detects Russell's Paradox in Frege's work. This sparks the foundational crisis.
- **1908:** L.E.J. Brouwer publishes arguments against the Law of Excluded Middle (LEM)
- **After 1913:** Brouwer dedicates himself to the development of intuitionism; he refuses a formalization of its logic.
- **1930s:** Heyting and Gentzen give formalizations of intuitionistic logic (as Hilbert and Natural Deduction systems respectively)
- **1943:** A. Heyting develops the Brouwer-Heyting-Kolmogorov (BHK)-interpretation of intuitionism
- **1956:** S. Kripke develops a semantics for propositional and first-order intuitionistic logic based on Kripke frames

Brouwer's Philosophy

- **Core Idea:** Mathematical objects and proofs have no existence independently of their construction.
- **Consequence:** Mathematics changes over time according to the corpus of assembled mathematical knowledge.
- This together with the separation of mathematics from language is the **first act of intuitionism**.
- **However:** Mathematics is not subjecive; Brouwer refers to an idealized creating subject

The BHK Interpretation

An interpretation of logical connectives and quantifiers in terms of proofs:

a proves A	conditions
$a : \bot$	false
$a : A \wedge B$	$a = (a_1, a_2)$, where $a_1 : A$ and $a_2 : B$
$a : A \vee B$	$a = (a_1, a_2)$, where $a_2 : A$ if $a_1 = 0$ and $a_2 : A$ if $a_1 = 1$
$a : A \to B$	for all p with $p : A$ we have $a(p) : B$
$a : \exists x A(x)$	$a = (a_1, a_2)$ and $a_2 : A(a_1)$
$a : \forall x A(x)$	for all $d \in D$ we have $a(d) : A(d)$, where D is given domain

Furthermore we define $\neg A :\equiv A \to \bot$.
The BHK interpretation is nicely captured by the Natural Deduction system on the board.

Semantics for classical first-order logic

A signature $\mathcal{R}$ is a set containing constant symbols, relational symbols and function symbols.

Definition

- An *$\mathcal{R}$-structure* is a tuple $\mathfrak{S} = (S, I_S)$ of a set S and a function $I_S : \mathcal{R} \to \bigcup_{n<\omega} S^n$ assigning constant symbols to elements of S, relation symbols to relations on S and function symbols to functions on S, respecting arity.
- For a first-order $\mathcal{R}$-sentence A with parameters in S we define the satisfaction relation $\mathfrak{S} \models A$ inductively as usual.
- For an $\mathcal{R}$-structure $\mathfrak{S}$ we denote by $\mathrm{Diag}^+(\mathfrak{S})$ its *positive diagram*, i.e., the set of all atomic $\mathcal{R}$-sentences A with parameters in S such that $\mathfrak{S} \models A$.

Kripke Models for **IQC** (1)

With **IQC** we refer to first-order Intuitionistic Calculus.

Definition

An **IQC**-*Kripke model* of signature $\mathcal{R}$ is a partial order $\langle W, \leq \rangle$ together with family $(\mathfrak{S}_w)_{w \in W}$ of $\mathcal{R}$-structures such that for all $v, w \in W$ with $v \leq w$ we have:

1. $S_v \subseteq S_w$;
2. the inclusion $i : \mathfrak{S}_v \to \mathfrak{S}_w$ is a homomorphism.

Heuristics: Partial order models flow of time. At every point there could be multiple possible futures – corresponding to knowledge not yet acquired.

Kripke Models for **IQC** (2)

Definition

For an **IQC**-Kripke model $\mathcal{K} = \langle W, \leq, (\mathfrak{S}_w)_{w \in W} \rangle$ of signature $\mathcal{R}$ we define a *Forcing relation* between nodes $w \in W$ and first-order $\mathcal{R}$-formulas with parameters in S_w as follows:

- $\mathcal{K}, w \Vdash A$ iff $A \in \mathrm{Diag}^+(\mathfrak{S}_w)$ for an atomic sentence A;
- $\mathcal{K}, w \nVdash \bot$;
- $\mathcal{K}, w \Vdash A \wedge B$ iff $\mathcal{K}, w \Vdash A$ and $\mathcal{K}, w \Vdash B$;
- $\mathcal{K}, w \Vdash A \vee B$ iff $\mathcal{K}, w \Vdash A$ or $\mathcal{K}, w \Vdash B$;
- $\mathcal{K}, w \Vdash A \rightarrow B$ iff for all $v \geq w$, $\mathcal{K}, v \Vdash A$ implies $\mathcal{K}, v \Vdash B$;
- $\mathcal{K}, w \Vdash (\forall x)A(x)$ iff for all $v \geq w$ and $d \in D_v$, $\mathcal{K}, v \Vdash A(d)$;
- $\mathcal{K}, w \Vdash (\exists x)A(x)$ iff there is $d \in D_w$ s.t. $\mathcal{K}, w \Vdash A(d)$.

Kripke Models for **IQC** (3)

Lemma (Upward persistency)

For any **IQC**-Kripke model $\mathcal{K}$, any nodes w, v with v, w and any atomic f.o. sentence of the right signature we have:

$$\mathcal{K}, v \Vdash A \quad \Rightarrow \mathcal{K}, w \Vdash A$$

Proof

Induction on the complexity of A using the fact that $S_v \subseteq S_w$ is an embedding from S_v into S_w and thus $\text{Diag}^+(\mathfrak{S}_v) \subseteq \text{Diag}^+(\mathfrak{S}_w)$.

Intuition: Once a fact is established it is never lost in the future.

Soundness and Completeness (1)

Definition

- For a **IQC**-Kripke model $\mathcal{K}$ we write $\mathcal{K} \models A$ to denote that for all nodes w in $\mathcal{K}$ we have $\mathcal{K}, w \Vdash A$.
- For a set Γ of first order sentences and a first order sentence A we write $\Gamma \models_{\mathbf{IQC}} A$ to mean that for every **IQC**-Kripke $\mathcal{K}$ we have:

$$\text{If for all } F \in \Gamma,\ \mathcal{K} \models F, \text{ then } \mathcal{K} \models A.$$

Soundness and Completeness (2)

Theorem (Soundness and Strong Completeness)

Let Γ be a set of first-order sentences and A a first-order sentence, then we have:

$$\Gamma \vdash_{\mathbf{IQC}} A \quad \Leftrightarrow \quad \Gamma \models_{\mathbf{IQC}} A.$$

Proof

Soundness: Induction on the length of the derivation $\Gamma \vdash_{\mathbf{IQC}} A$.
Completeness: Let Γ' be an **IQC**-consistent set of formulas. Combine the Henkin Construction from first-order logic with the Canonical Model Construction from Modal Logic to obtain a **IQC**-model $\mathcal{K}$ and a node w such that $\mathcal{K}, w \Vdash \Gamma'$.
Now assume that $\Gamma \nvdash_{\mathbf{IQC}} A$. Then $\Gamma \cup \{\neg A\}$ is consistent. Hence $\Gamma \cup \{\neg A\}$ is satisfiable. So $\Gamma \not\models_{\mathbf{IQC}} A$.

Soundness and Completeness (3)

Theorem (Tree Property)

Let Γ be a set of first-order sentences and A a first-order sentence, then if $\Gamma \nvdash_{\mathbf{IQC}} A$ there is already a Γ-model $\langle W, \leq, (D_w)_{w \in W} \rangle$ such that $\langle W, \leq \rangle$ is a tree with root r and $W, r \nVdash A$.

Proof

If $\Gamma \nvdash_{\mathbf{IQC}} A$, take by Completeness Γ-model $\mathcal{K}'$ s.t. $\mathcal{K}' \not\models A$.
Unravel $\mathcal{K}'$ to transform it into a model $\mathcal{K}''$ on a tree s.t. $\mathcal{K}'' \not\models A$.
Take r with $\mathcal{K}'', r \nVdash A$ and let $\mathcal{K}$ be the submodel generated by r.

Counterexamples

Kripke models give us the means to prove that certain principles are **not** intuitionistically valid.

Lemma

We have

- $\nvdash_{\mathbf{IQC}} A \vee \neg A$ for atomic A;
- $\nvdash_{\mathbf{IQC}} \neg(A \wedge B) \to (\neg A \vee \neg B)$ for atomic A, B;
- $\nvdash_{\mathbf{IQC}} \neg\forall x A(x) \to \exists x \neg A(x)$ for atomic $A(x)$;
- $\nvdash_{\mathbf{IQC}} \forall x(A \vee B(x)) \to A \vee \forall x B(x)$ for atomic A, $B(x)$.

Proof

See blackboard.

Heyting Arithmetic

Definition

We let Heyting Arithmetic (**HA**) be the intuitionistic theory generated by the Peano Axioms, i.e. the universal closures of

- $x + 1 \neq 0$,
- $x + 1 = y + 1 \to x = y$,
- $x + 0 = x$,
- $x + (y + 1) = (x + y) + 1$,
- $x \cdot 0 = 0$,
- $x \cdot (y + 1) = x \cdot y + x$,
- $A(0) \wedge (\forall x)[A(x) \to A(x + 1)] \to (\forall x)A(x)$ for any formula $A(x)$.

Disjunction Property and Existential Property for **HA** (1)

Lemma

Let $\mathcal{K}$ be a model of **HA**. Then there exists in the domain D_w of each world w a unique sequence of distinct elements that are the interpretations of the numerals $\underline{0}, \underline{1}, \ldots,$ where $\underline{0} = 0$ and $\underline{n+1} = \underline{n} + 1$.

Lemma (Smorynski's trick)

If M is a set of **HA**-models then a new **HA**-model is obtained by taking the disjoint union of M adding a new root w_0 below it such that $S_{w_0} = \omega$.

Disjunction Property and Existential Property for **HA** (2)

Theorem (DP and EP)

- **HA** has the Disjunction Property, i.e., **HA** $\vdash_{\mathbf{IQC}} A \vee B$ iff **HA** $\vdash_{\mathbf{IQC}} A$ or **HA** $\vdash_{\mathbf{IQC}} B$;
- **HA** has the Existential Property, i.e., **HA** $\vdash_{\mathbf{IQC}} \exists x A(x)$ iff **HA** $\vdash_{\mathbf{IQC}} A(\underline{n})$ for some $n \in \omega$.

Corollary

HA is strictly contained in **PA**.

de Jongh's Theorem

Let **IPC** denote the intuitionistic propositional calculus, which we can consider as a special case of **IQC**.

Theorem (de Jongh)

Let A be a propositional formula. Then $\vdash_{\mathbf{IPC}} A(p_1, \ldots, p_n)$ if and only if for all arithmetic sentences $\psi_1, \ldots, \psi_m$, $\mathbf{HA} \vdash_{\mathbf{IQC}} A(\psi_1, \ldots, \psi_n)$.