Proof Theory in The Light of Categories

Giovanni Ciná & Giuseppe Greco

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Part 1 - Proof Theory

1. **From global- to local-rules calculi**
   - Axiomatic Calculi
   - Natural Deduction Calculi
   - Sequent Calculi
   - Cut-elimination

2. **From holistic to modular calculi**
   - Display Calculi
   - Propositions- and Structures-Language
   - Display Postulates and Display Property
   - Structural Rules
   - Operational Rules
   - No-standard Rules

Axiomatic calculi *á la* Hilbert were the first to appear and, typically, are characterized by ‘more’ axioms and ‘few’ inference rules, at the limit only one (Modus Ponens).

- The objects manipulated in such calculi are ***formulas***.
- The meaning of logical symbols is ***implicitly defined*** by the axioms that, also, set their mutual relations.
- Again, the axioms allow only an ***indirect control*** of the ‘structure’.

1	$(A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$
2	$A \to ((A \to A) \to A)$
3	$(A \to (A \to A)) \to (A \to A)$
4	$A \to (A \to A)$
5	$A \to A$

$$\dfrac{\dfrac{1 \qquad 2}{3}\,\mathrm{MP} \qquad\qquad 4}{5}\,\mathrm{MP}$$

where the leaves are all instantiations of axioms.

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Advantages:

- proofs on the system are simplified for systems with few and simple inference rules;
- the space of logics can be reconstructed in a modular way: adding axioms to a previous axiomatization we get other logics.

Disadvantages:

- the proofs in the system are long and often unnatural;
- the meaning of connectives is global: e.g. the axiom $(A \to B) \to ((C \to B) \to (A \vee C \to B))$ involves different connectives;
- the derivations are global: e.g. only Modus Ponens is used to prove all theorems.

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Natural deduction calculi *á la* Gentzen are characterized by the use of assumptions (introduced by an explicit rule) and different inference rules for different connectives.

- The objects manipulated in such calculi are ***formulas***.
- The meaning of the logical symbols is ***explicitly defined*** (by Intr/Elim Rule): an operational content corresponds to each connective.
- Introduction Rules for implication and negation discharge assumptions: appropriate restrictions allow ***some control*** of the ‘structure’.

$$
\begin{prooftree}
\AxiomC{$[A \wedge B]^1$}
\AxiomC{$[\neg A \vee \neg B]^2$}
\AxiomC{$[A \wedge B]^3$}
\RightLabel{E$\wedge$}
\UnaryInfC{A}
\AxiomC{$[\neg A]^4$}
\RightLabel{I$\wedge$}
\BinaryInfC{$A \wedge \neg A$}
\RightLabel{3 I$\neg$}
\UnaryInfC{$\neg(A \wedge B)$}
\AxiomC{$[A \wedge B]^5$}
\RightLabel{E$\wedge$}
\UnaryInfC{B}
\AxiomC{$[\neg B]^6$}
\RightLabel{I$\wedge$}
\BinaryInfC{$B \wedge \neg B$}
\RightLabel{5 I$\neg$}
\UnaryInfC{$\neg(A \wedge B)$}
\RightLabel{4,6 E$\vee$}
\TrinaryInfC{$\neg(A \wedge B)$}
\RightLabel{I$\wedge$}
\BinaryInfC{$(A \wedge B) \wedge \neg(A \wedge B)$}
\RightLabel{2 I$\neg$}
\UnaryInfC{$\neg(\neg A \vee \neg B)$}
\RightLabel{1,3,5 I$\rightarrow$}
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Advantages:

- the proofs in the system are natural;
- the connectives are introduced one by one (this is in the direction of proof-theoretic semantics);

Disadvantages:

- assumptions tipically are discharged after many steps in a derivation;
- it is not simple to reconstruct the space of the logics;
- it is difficult to obtain natural deduction calculi for non-classical or modal logics.

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Sequent calculi *á la* Gentzen are characterized by a single axiom (Identity), the use of assumptions and conclusions, by different inference rules for different connectives and for different structural operations.

- Objects manipulated in such calculations are ***sequents***: $\Gamma \vdash \Delta$ where Γ and Δ are (possibly empty) sequences of formulas separated by a (poliadyc) comma.
- The meaning of logical symbols is ***explicitly defined*** (by Left/Right Introduction Rule).
- The structural rules allow a ***direct control*** of the 'structure'.

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Advantages:

- the derivations are local;
- the proofs in the system are ***automatizable*** (if the calculus enjoy cut-elimination);
- a distinction between connectives and structure is introduced (this is in the direction of proof-theoretic semantics).

Disadvantages:

- the space of logics cannot be reconstructed in a modular way (if the calculus is non-standard, i.e. as usual for modal logics);
- it is not simple to obtain sequent calculi for substructural or modal logics (with the sub-formula property).

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Common forms of the cut rule are the following:

$$\frac{\Gamma \vdash C, \Delta \qquad \Gamma', C \vdash \Delta'}{\Gamma', \Gamma \vdash \Delta', \Delta} \qquad \frac{\Gamma \vdash C, \Delta \qquad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash C \qquad \Gamma', C \vdash \Delta}{\Gamma', \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash C, \Delta \qquad C \vdash \Delta'}{\Gamma \vdash \Delta', \Delta}$$

Theorem (Cut-elimination)

If $\Gamma \vdash \Delta$ is derivable in the calculus S with Cut, then it is in S without Cut.

The ***cut-elimination*** is the most fundamental technique in proof theory and many important syntactic properties derive from it (e.g. ***decidability***).

A cut is an intermediate step in a deduction, by which a conclusion(s) Δ can be proved from the assumption(s) Γ via the lemma C.

'Eliminating the cut' from such a proof generates a new (and lemma-free) proof of Δ, which exclusively employs syntactic material coming from Γ and Δ (***subformula property***).

Typically, syntactic proofs of cut-elimination are non-modular: if a new rule is added, cut-elimination must be proved from scratch.

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Display calculi were introduced by Belnap **[1.2]** to provide a

- uniform account for cut-elimination;
- a ‘pure’ proof-theoretical analysis of logics;
- a tool useful to ‘merge’ different logics.

Display calculi generalize sequent calculi allowing:

- different ‘***structural connectives***’ (not just the Gentzen’s comma), where the structures in $X \vdash Y$ are binary trees (not sequences);
- a set of structural rules named ***Display Postulates***, that give the Display Property (essential in Belnap’s cut-elimination).

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Advantages:

- cut-elimination is a consequence of design principles, by the following:

Theorem (Cut-elimination [1.2] [1.5])

If a logic is 'properly displayable', then it enjoys cut-elimination

- space of logics can be reconstructed in a modular way, because of:

Došen Principle [1.5]

The rules for the logical operations are never changed: all changes are made in the structural rules

- a 'real' proof-theory is possible for substrucural and modal logics (e.g. ***separated***, ***symmetrical*** and ***explicit*** introduction rules for the normal modal operators are available).

Disadvantages:

- not amenable for proof-search (because of Display Postulates).

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As case study, we consider the display calculus (plus explicit negations) introduced in Greco, Kurz, Palmigiano **[1.3]** for the Baltag-Moss-Solecki logic of Epistemic Actions and Knowledge EAK **[1.1]**.

For each agent $\mathsf{a} \in \mathsf{Ag}$ and action $\alpha \in \mathsf{Act}$,

- ***Propositions*** are built from a set of atomic propositional variables $\mathsf{AtProp} = \{p, q, r, \ldots\}$ and two constants $\bot$ and $\top$:

$$A := \begin{cases} p \mid \bot \mid \top \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid A \succ\!\!- A \mid \neg A \mid {\sim} A \mid \\ \Diamond_{\mathsf{a}} A \mid \Box_{\mathsf{a}} A \mid \blacklozenge_{\mathsf{a}} A \mid \blacksquare_{\mathsf{a}} A \mid [\alpha] A \mid \langle \alpha \rangle A \mid \widehat{\alpha} A \mid \overline{\underline{\alpha}} A\,. \end{cases}$$

- ***Structures*** are built from formulas and one structural constant I:

$$X := \begin{cases} \mathrm{I} \mid A \mid X ; X \mid X > X \mid {*}X \mid \\ \bullet_{\mathsf{a}} X \mid \circ_{\mathsf{a}} X \mid \{\alpha\} X \mid \overbrace{\underbrace{\alpha}} X\,. \end{cases}$$

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For each agent a $\in$ Ag and action $\alpha \in$ Act,

- ***Propositions*** are built from a set of atomic propositional variables AtProp $= \{p, q, r, \ldots\}$ and two constants $\bot$ and $\top$:

$$A := \begin{cases} p \mid \bot \mid \top \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid A \succ\!\!-A \mid \neg A \mid {\sim}A \mid \\ \Diamond_{\mathrm{a}} A \mid \Box_{\mathrm{a}} A \mid \blacklozenge_{\mathrm{a}} A \mid \blacksquare_{\mathrm{a}} A \mid [\alpha] A \mid \langle\alpha\rangle A \mid \widehat{\underline{\alpha}} A \mid \overline{\underline{\alpha}} A\,. \end{cases}$$

- ***Structures*** are built from formulas and one structural constant I:

$$X := \begin{cases} \mathrm{I} \mid A \mid X ; X \mid X > X \mid {*}X \mid \\ \bullet_{\mathrm{a}} X \mid \circ_{\mathrm{a}} X \mid \{\alpha\} X \mid \overbrace{\underbrace{\alpha}} X\,. \end{cases}$$

As case study, we consider the display calculus (plus explicit negations) introduced in Greco, Kurz, Palmigiano **[1.3]** for the Baltag-Moss-Solecki logic of Epistemic Actions and Knowledge EAK **[1.1]**.

For each agent $\mathsf{a} \in \mathsf{Ag}$ and action $\alpha \in \mathsf{Act}$,

- ***Propositions*** are built from a set of atomic propositional variables $\mathsf{AtProp} = \{p, q, r, \ldots\}$ and two constants $\bot$ and $\top$:

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- ***Structures*** are built from formulas and one structural constant I:

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The structural connectives are ***contextual*** (as the Gentzen's comma) and each of them is associated with a pair of logical connectives:

Structural symb:	I		$;$		$>$		$*$	
Operational symb:	$\top$	$\bot$	$\wedge$	$\vee$	$\succ\!\!-$	$\rightarrow$	$\neg$	$\sim$

Structural symb:	$\circ_{\mathrm{a}}$		$\bullet_{\mathrm{a}}$		$\{\alpha\}$		$\overbrace{\underbrace{\alpha}}$	
Operational symb:	$\Diamond_{\mathrm{a}}$	$\Box_{\mathrm{a}}$	$\blacklozenge_{\mathrm{a}}$	$\blacksquare_{\mathrm{a}}$	$\langle\alpha\rangle$	$[\alpha]$	$\widehat{\widecheck{\alpha}}$	$\overline{\underline{\alpha}}$

by the translations τ_1 of precedent and τ_2 of succedent into prop. :

$$
\begin{array}{rcl@{\qquad}rcl}
\tau_1(A) & := & A & \tau_2(A) & := & A \\
\tau_1(\mathrm{I}) & := & \top & \tau_2(\mathrm{I}) & := & \bot \\
\tau_1(X\,;Y) & := & \tau_1(X)\wedge\tau_1(Y) & \tau_2(X\,;Y) & := & \tau_2(X)\vee\tau_2(Y) \\
\tau_1(X>Y) & := & \tau_2(X)\succ\!\!-\tau_1(Y) & \tau_2(X>Y) & := & \tau_1(X)\rightarrow\tau_2(Y) \\
\tau_1(*X) & := & \sim\tau_2(X) & \tau_2(*X) & := & \neg\tau_1(X) \\
\tau_1(\circ_{\mathrm{a}}X) & := & \Diamond_{\mathrm{a}}\tau_1(X) & \tau_2(\circ_{\mathrm{a}}X) & := & \Box_{\mathrm{a}}\tau_2(X) \\
\tau_1(\bullet_{\mathrm{a}}X) & := & \blacklozenge_{\mathrm{a}}\tau_1(X) & \tau_2(\bullet_{\mathrm{a}}X) & := & \blacksquare_{\mathrm{a}}\tau_2(X) \\
\tau_1(\{\alpha\}X) & := & \langle\alpha\rangle\tau_1(X) & \tau_2(\{\alpha\}X) & := & \langle\alpha\rangle\tau_2(X) \\
\tau_1(\overbrace{\underbrace{\alpha}}X) & := & \widehat{\widecheck{\alpha}}\,\tau_1(X) & \tau_2(\{\alpha\}X) & := & \overline{\underline{\alpha}}\,\tau_2(X)
\end{array}
$$

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Structural symb:	I		$;$		$>$		$*$	
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Structural symb:	$\circ_a$		$\bullet_a$		$\{\alpha\}$		$\overbrace{\underbrace{\alpha}}$	
Operational symb:	$\Diamond_a$	$\Box_a$	$\blacklozenge_a$	$\blacksquare_a$	$\langle\alpha\rangle$	$[\alpha]$	$\widehat{\widecheck{\alpha}}$	$\overline{\underline{\alpha}}$

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\tau_1(*X) & := & \sim\tau_2(X) & \tau_2(*X) & := & \neg\tau_1(X) \\
\tau_1(\circ_a X) & := & \Diamond_a \tau_1(X) & \tau_2(\circ_a X) & := & \Box_a \tau_2(X) \\
\tau_1(\bullet_a X) & := & \blacklozenge_a \tau_1(X) & \tau_2(\bullet_a X) & := & \blacksquare_a \tau_2(X) \\
\tau_1(\{\alpha\}X) & := & \langle\alpha\rangle\tau_1(X) & \tau_2(\{\alpha\}X) & := & \langle\alpha\rangle\tau_2(X) \\
\tau_1(\overbrace{\underbrace{\alpha}}X) & := & \widehat{\widecheck{\alpha}}\,\tau_1(X) & \tau_2(\{\alpha\}X) & := & \overline{\underline{\alpha}}\,\tau_2(X)
\end{array}
$$

Display Postulates

$$\begin{array}{c} ; \\ > \end{array} \frac{X ; Y \vdash Z}{Y \vdash X > Z} \qquad \frac{Z \vdash Y ; X}{Y > Z \vdash X} \begin{array}{c} > \\ ; \end{array}$$

$$\begin{array}{c} \circ_a \\ \bullet_a \end{array} \frac{\circ_a X \vdash Y}{X \vdash \bullet_a Y} \qquad \frac{X \vdash \circ_a Y}{\bullet_a X \vdash Y} \begin{array}{c} \bullet_a \\ \circ_a \end{array}$$

$$\begin{array}{c} \{\alpha\} \\ \overbrace{\underbrace{\alpha}} \end{array} \frac{\{\alpha\} X \vdash Y}{X \vdash \overbrace{\underbrace{\alpha}} Y} \qquad \frac{X \vdash \{\alpha\} Y}{\overbrace{\underbrace{\alpha}} X \vdash Y} \begin{array}{c} \overbrace{\underbrace{\alpha}} \\ \{\alpha\} \end{array}$$

$$\begin{array}{c} * \\ *_L \end{array} \frac{*X \vdash Y}{*Y \vdash X} \qquad \frac{Y \vdash *X}{X \vdash *Y} \begin{array}{c} * \\ *_R \end{array}$$

$$\begin{array}{c} ;* \\ ; \end{array} \frac{Z \vdash Y ; X}{*Y ; Z \vdash X} \qquad \frac{X ; Y \vdash Z}{Y \vdash *X ; Z} \begin{array}{c} ; \\ ;* \end{array}$$

$$**_L \frac{**X \vdash Y}{X \vdash Y} \qquad \frac{Y \vdash **X}{Y \vdash X} **_R$$

Display Postulates

$$
\begin{array}{c} ; \\ > \end{array}\ \begin{array}{c} X ; Y \vdash Z \\ \hline\hline Y \vdash X > Z \end{array}
\qquad
\begin{array}{c} Z \vdash Y ; X \\ \hline\hline Y > Z \vdash X \end{array}\ \begin{array}{c} > \\ ; \end{array}
$$

$$
\begin{array}{c} \circ_a \\ \bullet_a \end{array}\ \begin{array}{c} \circ_a X \vdash Y \\ \hline\hline X \vdash \bullet_a Y \end{array}
\qquad
\begin{array}{c} X \vdash \circ_a Y \\ \hline\hline \bullet_a X \vdash Y \end{array}\ \begin{array}{c} \bullet_a \\ \circ_a \end{array}
$$

$$
\begin{array}{c} \{\alpha\} \\ \underbrace{\overbrace{\alpha}} \end{array}\ \begin{array}{c} \{\alpha\} X \vdash Y \\ \hline\hline X \vdash \underbrace{\overbrace{\alpha}} Y \end{array}
\qquad
\begin{array}{c} X \vdash \{\alpha\} Y \\ \hline\hline \underbrace{\overbrace{\alpha}} X \vdash Y \end{array}\ \begin{array}{c} \underbrace{\overbrace{\alpha}} \\ \{\alpha\} \end{array}
$$

$$
\begin{array}{c} * \\ *_L \end{array}\ \begin{array}{c} *X \vdash Y \\ \hline\hline *Y \vdash X \end{array}
\qquad
\begin{array}{c} Y \vdash *X \\ \hline\hline X \vdash *Y \end{array}\ \begin{array}{c} * \\ *_R \end{array}
$$

$$
\begin{array}{c} ;* \\ ; \end{array}\ \begin{array}{c} Z \vdash Y ; X \\ \hline\hline *Y ; Z \vdash X \end{array}
\qquad
\begin{array}{c} X ; Y \vdash Z \\ \hline\hline Y \vdash *X ; Z \end{array}\ \begin{array}{c} ; \\ ;* \end{array}
$$

$$
_L\ \frac{X \vdash Y}{X \vdash Y}
\qquad
\frac{Y \vdash **X}{Y \vdash X}\ **_R
$$

By definition, structural connectives form ***adjoint pairs*** as follows:

$$; \dashv > \qquad > \dashv ; \qquad \circ_a \dashv \bullet_a \qquad \bullet_a \dashv \circ_a \qquad * \dashv *$$

(1) Nota Bene: ‘adjointness’ in Part 2.

So, Display Postulates are ‘about the connection between left and right side of the turnstile’.

By definition, structural connectives form ***adjoint pairs*** as follows:

$$; \dashv > \qquad > \dashv ; \qquad \circ_a \dashv \bullet_a \qquad \bullet_a \dashv \circ_a \qquad * \dashv *$$

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So, Display Postulates are ‘about the connection between left and right side of the turnstile’.

The Display Postulates allow to disassembly and reassembly structures and provide the following:

Theorem (Display Property [1.2] [1.5])

Each substructure in a display-sequent is isolable or 'displayable' in precedent or, exclusively, succedent position.

Note that 'in precedent (succedent) position' and 'on the left (right) side of turnstile' coincide in a Gentzen's sequent calculus, but not in a display calculus. E.g. In '$Y \vdash X > Z$', X is on the right of the turnstile but it is in a precedent position, in fact it is displayable on the left side:

$$\cfrac{\cfrac{\cfrac{Y \vdash X > Z}{X \,;\, Y \vdash Z}}{Y \,;\, X \vdash Z}}{X \vdash Y > Z}$$

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$$\cfrac{\cfrac{\cfrac{Y \vdash X > Z}{X \,;\, Y \vdash Z}}{Y \,;\, X \vdash Z}}{X \vdash Y > Z}$$

Let be $\odot \in \{\circ_{\mathrm{a}}, \bullet_{\mathrm{a}}\}$.

Structural Rules

- entry/exit rules -

$$Id\ \frac{}{p \vdash p} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ Cut$$

$$\mathrm{I}_L\ \frac{X \vdash Y}{X \,;\, \mathrm{I} \vdash Y} \qquad \frac{Y \vdash X}{Y \vdash \mathrm{I} \,;\, X}\ \mathrm{I}_R$$

$$W_L\ \frac{X \vdash Z}{X \,;\, Y \vdash Z} \qquad \frac{Z \vdash Y}{Z \vdash Y \,;\, X}\ W_R$$

$$C_L\ \frac{X \,;\, X \vdash Y}{X \vdash Y} \qquad \frac{Y \vdash X \,;\, X}{Y \vdash X}\ C_R$$

$$\overset{\odot}{\mathrm{I}}\ \frac{X \vdash \mathrm{I}}{\odot X \vdash \mathrm{I}} \qquad \frac{\mathrm{I} \vdash X}{\mathrm{I} \vdash \odot X}\ \underset{\odot}{\mathrm{I}}$$

$$\overset{*}{\mathrm{I}}\ \frac{\mathrm{I} \vdash X}{*X \vdash \mathrm{I}} \qquad \frac{X \vdash \mathrm{I}}{\mathrm{I} \vdash *X}\ \underset{*}{\mathrm{I}}$$

Let be $\circledast \in \{*, \circ_{\mathsf{a}}, \bullet_{\mathsf{a}}, \{\alpha\}, \overbrace{\underbrace{\alpha}}\}$.

$$
\frac{\circledast X \,;\, \circledast Y \vdash Z}{\circledast(X \,;\, Y) \vdash Z}\;{\scriptstyle \circledast ;}
\qquad
\frac{Z \vdash \circledast Y \,;\, \circledast X}{Z \vdash \circledast(Y \,;\, X)}\;{\scriptstyle ; \circledast}
$$

$$
\frac{\circledast X > \circledast Y \vdash Z}{\circledast(X > Y) \vdash Z}\;{\scriptstyle \circledast >}
\qquad
\frac{Z \vdash \circledast Y > \circledast X}{Z \vdash \circledast(Y > X)}\;{\scriptstyle > \circledast}
$$

- manipulation rules -

$$
E_L\;\frac{Y \,;\, X \vdash Z}{X \,;\, Y \vdash Z}
\qquad
\frac{Z \vdash X \,;\, Y}{Z \vdash Y \,;\, X}\;E_R
$$

$$
A_L\;\frac{X \,;\, (Y \,;\, Z) \vdash W}{(X \,;\, Y) \,;\, Z \vdash W}
\qquad
\frac{W \vdash (Z \,;\, Y) \,;\, X}{W \vdash Z \,;\, (Y \,;\, X)}\;A_R
$$

$$
Grn_L\;\frac{X > (Y \,;\, Z) \vdash W}{\overline{(X > Y) \,;\, Z \vdash W}}
\qquad
\frac{W \vdash X > (Y \,;\, Z)}{\overline{W \vdash (X > Y) \,;\, Z}}\;Grn_R
$$

(2) Nota Bene: ‘naturality’ in Part 2.

So, Structural Rules are ‘about the left side or, exclusively, the right side of the turnstile’.

Note that the Excluded Middle is derivable by Grishin’s rules as follows:

$$
\begin{array}{l}
A \vdash A \\ \hline
A \,;\, \mathrm{I} \vdash A \\ \hline
A \,;\, \mathrm{I} \vdash \bot \,;\, A \\ \hline
I \vdash A > (\bot \,;\, A) \\ \hline
\mathrm{I} \vdash (A > \bot) \,;\, A \quad \textit{Grn} \\ \hline
\mathrm{I} \vdash A \,;\, (A > \bot) \\ \hline
A > \mathrm{I} \vdash A > \bot \\ \hline
A > \mathrm{I} \vdash A \to \bot \\ \hline
A > \mathrm{I} \vdash \neg A \\ \hline
\mathrm{I} \vdash A \,;\, \neg A \\ \hline
\mathrm{I} \vdash A \vee \neg A
\end{array}
$$

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So, Structural Rules are ‘about the left side or, exclusively, the right side of the turnstile’.

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$$
\begin{array}{c}
A \vdash A \\
\hline
A \,;\mathrm{I} \vdash A \\
\hline
A \,;\mathrm{I} \vdash \bot \,; A \\
\hline
I \vdash A > (\bot \,; A) \\
\hline
\mathrm{I} \vdash (A > \bot) \,; A \\
\hline
\mathrm{I} \vdash A \,; (A > \bot) \\
\hline
A > \mathrm{I} \vdash A > \bot \\
\hline
A > \mathrm{I} \vdash A \to \bot \\
\hline
A > \mathrm{I} \vdash \neg A \\
\hline
\mathrm{I} \vdash A \,; \neg A \\
\hline
\mathrm{I} \vdash A \vee \neg A
\end{array}
$$

Grn

Operational Rules

- translation rules -

$$\bot_L \ \frac{}{\bot \vdash \mathrm{I}} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \bot} \ \bot_R$$

$$\top_L \ \frac{\mathrm{I} \vdash X}{\top \vdash X} \qquad \frac{}{\mathrm{I} \vdash \top} \ \top_R$$

$$\wedge_L \ \frac{A \,;\, B \vdash Z}{A \wedge B \vdash Z} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \,;\, Y \vdash A \wedge B} \ \wedge_R$$

$$\vee_L \ \frac{B \vdash Y \qquad A \vdash X}{B \vee A \vdash Y \,;\, X} \qquad \frac{Z \vdash B \,;\, A}{Z \vdash B \vee A} \ \vee_R$$

$$\rightarrow_L \ \frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash X > Y} \qquad \frac{Z \vdash A > B}{Z \vdash A \rightarrow B} \ \rightarrow_R$$

$$\succ\!\!-_L \ \frac{A > B \vdash Z}{A \succ\!\!- B \vdash Z} \qquad \frac{Y \vdash B \qquad A \vdash X}{X > Y \vdash A \succ\!\!- B} \ \succ\!\!-_R$$

Let be $\odot_\alpha \in \{\circ_{\mathrm{a}}, \bullet_{\mathrm{a}}, \{\alpha\}, \overbrace{\underbrace{\alpha}}\}$,

$\dot{\Diamond}_\alpha \in \{\Diamond_{\mathrm{a}}, \blacklozenge_{\mathrm{a}}, \langle\alpha\rangle, \widehat{\underline{\alpha}}\}$,

$\boxdot_\alpha \in \{\Box_{\mathrm{a}}, \blacksquare_{\mathrm{a}}, [\alpha], \overline{\underline{\alpha}}\}$.

$$\dot{\Diamond}_{\alpha L}\ \frac{\odot_\alpha A \vdash X}{\dot{\Diamond}_\alpha A \vdash X} \qquad \frac{X \vdash A}{\odot_\alpha X \vdash \dot{\Diamond}_\alpha A}\ \dot{\Diamond}_{\alpha R}$$

$$\boxdot_{\alpha L}\ \frac{A \vdash X}{\boxdot_\alpha A \vdash \odot_\alpha X} \qquad \frac{X \vdash \odot_\alpha A}{X \vdash \boxdot_\alpha A}\ \boxdot_{\alpha R}$$

$$\sim_L\ \frac{*A \vdash X}{\sim A \vdash X} \qquad \frac{A \vdash X}{*X \vdash \sim A}\ \sim_R$$

$$\neg_L\ \frac{X \vdash A}{\neg A \vdash *X} \qquad \frac{X \vdash *A}{X \vdash \neg A}\ \neg_R$$

(3) Nota Bene: '*functoriality*' in Part 2.

So, (one half of the) Operational Rules are 'about left and right side of the turnstile at the same time'.

Let be $\odot_\alpha \in \{\circ_{\mathsf{a}}, \bullet_{\mathsf{a}}, \{\alpha\}, \overbrace{\underbrace{\alpha}}\}$,

$\Diamond\!\!\!\cdot_\alpha \in \{\Diamond_{\mathsf{a}}, \blacklozenge_{\mathsf{a}}, \langle\alpha\rangle, \widehat{\underset{\smile}{\alpha}}\}$,

$\boxdot_\alpha \in \{\Box_{\mathsf{a}}, \blacksquare_{\mathsf{a}}, [\alpha], \overline{\underline{\alpha}}\}$.

$$\Diamond\!\!\!\cdot_{\alpha L}\ \frac{\odot_\alpha A \vdash X}{\Diamond\!\!\!\cdot_\alpha A \vdash X} \qquad \frac{X \vdash A}{\odot_\alpha X \vdash \Diamond\!\!\!\cdot_\alpha A}\ \Diamond\!\!\!\cdot_{\alpha R}$$

$$\boxdot_{\alpha L}\ \frac{A \vdash X}{\boxdot_\alpha A \vdash \odot_\alpha X} \qquad \frac{X \vdash \odot_\alpha A}{X \vdash \boxdot_\alpha A}\ \boxdot_{\alpha R}$$

$$\sim_L\ \frac{*A \vdash X}{\sim A \vdash X} \qquad \frac{A \vdash X}{*X \vdash \sim A}\ \sim_R$$

$$\neg_L\ \frac{X \vdash A}{\neg A \vdash *X} \qquad \frac{X \vdash *A}{X \vdash \neg A}\ \neg_R$$

(3) Nota Bene: ‘*functoriality*’ in Part 2.

So, (one half of the) Operational Rules are ‘about left and right side of the turnstile at the same time’.

In a context whit $Pre(\alpha)$, we allow the following no-standard rules.

Contextual Operational Rules

- translation rules -

$$\mathit{reverse}_L \ \frac{Pre(\alpha)\, ; \{\alpha\}A \vdash X}{Pre(\alpha)\, ; [\alpha]A \vdash X} \qquad \frac{X \vdash Pre(\alpha) > \{\alpha\}A}{X \vdash Pre(\alpha) > \langle\alpha\rangle A}\ \mathit{reverse}_R$$

Contextual Structural Rules

- entry/exit rules -

$$\frac{X \vdash Y}{\{\alpha\}X \vdash \{\alpha\}Y}\ \textit{balance}$$

$$\textit{atom}_L\ \frac{}{\{\alpha\}\,p \vdash p} \qquad \frac{}{p \vdash \{\alpha\}\,p}\ \textit{atom}_R$$

$$\textit{reduce}_L\ \frac{Pre(\alpha)\,;\{\alpha\}A \vdash X}{\{\alpha\}A \vdash X} \qquad \frac{X \vdash Pre(\alpha) > \{\alpha\}A}{X \vdash \{\alpha\}A}\ \textit{reduce}_R$$

- manipulation rules -

$$\textit{swap-in}_L\ \frac{Pre(\alpha)\,;\{\alpha\}\circ_a X \vdash Y}{Pre(\alpha)\,;\circ_a\{\beta\}_{\alpha a \beta}\,X \vdash Y} \qquad \frac{Y \vdash Pre(\alpha) > \{\alpha\}\circ_a X}{Y \vdash Pre(\alpha) > \circ_a\{\beta\}_{\alpha a \beta}\,X}\ \textit{swap-in}_R$$

$$\textit{s-out}_L\ \frac{\Big(Pre(\alpha)\,;\circ_a\{\beta\}\,X \vdash Y \mid \alpha a \beta\Big)}{Pre(\alpha)\,;\{\alpha\}\circ_a X \vdash \mathbf{;}\Big(Y \mid \alpha a \beta\Big)} \qquad \frac{\Big(Y \vdash Pre(\alpha) > \circ_a\{\beta\}\,X \mid \alpha a \beta\Big)}{\mathbf{;}\Big(Y \mid \alpha a \beta\Big) \vdash Pre(\alpha) > \{\alpha\}\circ_a X}\ \textit{s-out}_R$$

[1.1] A. Baltag, L.S. Moss, S. Solecki, ***The logic of public announcements, common knowledge and private suspicions***, TARK, 43-56, **1998**

[1.2] N. Belnap, ***Display logic***, Journal of Philosophical Logic, 11: 375-417, **1982**

[1.3] G. Greco, A. Kurz, A. Palmigiano, ***Dynamic Epistemic Logic Displayed***, Submitted, **2013**.

[1.4] R. Goré, L. Postniece, A. Tiu, ***Cut-elimination and Proof Search for Bi-Intuitionistic Tense Logic***, Proc. Adv. in Modal Logic, 156-177, **2010**

[1.5] H. Wansing, ***Displaying modal logic***, Kluwer Academic Publishers, **1998**

Outline

Part 2 - Category Theory

Beware: we will be sloppy and intuitive on the technical details.
Main reference: S. Awodey. *Category Theory*, Oxford Logic Guides, vol. 49. Oxford: Oxford University Press, 2006.

Categories and functors

Definition

A *category* **C** is made of

- objects $A, B, C, \ldots$
- arrows $f : A \to B, g : A \to C, \ldots$

Arrows are closed under composition (when target and source match) and composition of arrows is associative. Every object A has an identity arrow 1_A that works as the unit of the composition.

Definition

A *functor* $F : \mathbf{C} \to \mathbf{D}$ is a pair of maps (F_1, F_2) such that

- F_1 maps object of **C** in objects of **D**
- F_1 maps arrows of **C** in arrows of **D**

and also preserves sources and targets, identities and compositions.

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- objects $A, B, C, \dots$
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and also preserves sources and targets, identities and compositions.

Definition

Given **C** and **D**, the *product category* $\mathbf{C} \times \mathbf{D}$ has

- as objects pairs of objects (C, D), with C in **C** and D in **D**
- as arrows pairs of arrows (f, f'), with f in **C** and f' in **D**

In a category **B** the *product* of two objects A, B is an object $A \times B$ equipped with two arrows $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ (projections) such that $\forall C, f_1 : C \to A, f_2 C \to B \; \exists! g : C \to A \times B$ that makes the following diagram commute

$$\begin{tikzcd}
 & C \arrow[dl, "f_1"'] \arrow[d, "g"] \arrow[dr, "f_2"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

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- as arrows pairs of arrows (f, f'), with f in **C** and f' in **D**

In a category **B** the *product* of two objects A, B is an object $A \times B$ equipped with two arrows $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ (projections) such that $\forall C, f_1 : C \to A, f_2 C \to B \, \exists! g : C \to A \times B$ that makes the following diagram commute

$$\begin{tikzcd} & C \arrow[dl, "f_1"'] \arrow[d, "g"] \arrow[dr, "f_2"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

Natural transformations

Definition

Given two functors $F, G : \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\eta : F \to G$ is a family of arrows in **D** indexed by the objects of **C** such that, for every arrow $f : C \to B$ in **C**, in **D** we have

$$\begin{array}{ccc} F(C) & \xrightarrow[F(f)]{} & F(B) \\ \downarrow{\scriptstyle \eta_C} & & \downarrow{\scriptstyle \eta_B} \\ G(C) & \xrightarrow{G(f)} & G(B) \end{array}$$

If all the arrows in the family η are isomorphisms, we call η a *natural isomorphism*.

Adjoints

Definition

Given two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ we say that F *is left adjoint of* G, in symbols $F \dashv G$, if $\forall C$ in $\mathbf{C}$ and D in $\mathbf{D}$ there is a bijective correspondence between arrows $F(C) \to D$ in $\mathbf{D}$ and arrows $C \to G(B)$ in $\mathbf{C}$.

This is usually written

$$\frac{F(C) \to D}{\overline{C \to G(B)}}$$

Moreover, this bijection is natural both in C and D.

Why do we need all this? The core idea is the following:

display calculi are calculi for arrows

in the sense that *proofs are seen as arrows*,

$$\frac{A \vdash_1 B}{C \vdash_2 D}$$

means *there is a unique way to build* $\vdash_2$ *from* $\vdash_1$ and

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We have seen that in Display Calculi we have Operational Rules, Structural Rules and Display Rules; each having a particular shape. The shapes are, respectively

$$1.\ \frac{C \vdash D}{F(C) \vdash F(D)} \qquad 2.\ \frac{C \vdash F(D)}{C \vdash G(D)} \qquad 3.\ \frac{F(C) \vdash D}{\overline{\overline{C \vdash G(D)}}}$$

When we read these as instructions to build arrows, we have that

- rules of type 1 are given by *functoriality*
- rules of type 2 are given by *naturality*
- rules of type 3 are given by *adjointness*

(Note: we also have rules of type 2 with functors on the left)

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We have seen the general idea, let us try to be more precise.

Definition

A *logic* is a category **U** having categories as objects and functors as arrows; one of such object is called **P**, it is the category having formulas as objects and proofs as arrows.

This seems a rather abstract and weak definition. However, this already ensure that, being **P** a category,

1. there is a proof $A \vdash_{Id_A} A$ for each formula A, the identity proof
2. we can compose proofs if target and source match, the Cut

$$\frac{A \vdash_x B \qquad B \vdash_y C}{A \vdash_{y \circ x} C}$$

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In this framework the features of a logic depend on the following additional assumptions:

- the objects in **U** and the closure under specific categorical constructions (e.g. the product) determine the “sorts”of the logic
- the arrows in **U** determines the connectives of the logic
- the relations between these arrows (seen as functors), give the rules of the logic

Minimal Logic

To see how this work we will look at an example: Minimal Logic. Assume **U** contains P and is closed under products, terminal objects and $()^{op}$, the functor that flips all the arrows in the category.

Assume the existence of the functors:

- $\wedge : \mathbf{P} \times \mathbf{P} \to \mathbf{P}$
- $\vee : \mathbf{P}^{op} \times \mathbf{P}^{op} \to \mathbf{P}^{op}$
- $\to : \mathbf{P}^{op} \times \mathbf{P} \to \mathbf{P}$
- $\neg : \mathbf{P}^{op} \to \mathbf{P}$
- $\top : 1 \to \mathbf{P}$

Note that we cannot just define these functors: we need to assume the existence of enough arrows in **P** in order to have functoriality.

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In order to have the usual rules for Minimal Logic, we need to buy something more, namely the mutual relations between such functors. We assume the adjunctions

- $\vee \dashv \Delta \dashv \wedge$
- $A \wedge \dashv A \rightarrow$, for all formulas A in **P**
- $\top \wedge \dashv Id$ (Id is the identity functor)

and the natural isomorphism (here displayed in the components A, B)

- $(A \rightarrow B) \wedge (A \rightarrow \neg B) \simeq \neg A$

These give us the usual rules of minimal logic (examples at the blackboard).

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Advantages

The framework we sketched has the following advantages:

- *Modularity*: Each connective is introduced in isolation, and we are extremely free in drawing the mutual relations between the connectives (but we are constrained by the categorical contructions available in **U**).
- *Behaviourism*: We do not define a language starting from atomic propositions, we merely define possible constructions on formulas (functors) and their effects on proofs (adjoints, nat. transformations).
- *Cut Elimination*: Being close to display calculi, it is easy in principle to check the premises of Belnap's Theorem.
- *Reduction*: If the project is feasible, the framework allows for a completely categorical treatment of syntax and proof theory.

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Future work

Some ideas for what we will do next:

- Explore the capabilities of the framework in expressing more complex logics, such as modal logic or first order logic.
- Investigate how this work relates to Categorical Semantics and Functorial Model Theory.
- Study how the framework is connected with the other works in Categorical Proof Theory and the issue of identity of proofs.
- Can the framework give insights into Proof Theoretic Semantics?