Boxing with Your Feet

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"Taking the Principle of the Excluded Middle from the mathematician ... is the same as ... prohibiting the boxer the use of his fists"
David Hilbert

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Foreword

This handout gives a brief account of a few aspects of Brouwer's mathematics of the continuum, exploring in particular his intuitionistic reconstruction of mathematics through his views on the continuum. It is a work in progress, given how much could be said about analysis and combinatorics in a constructive setting, and we may return to at a later stage. In particular, it would be quite interesting to elaborate on the contemporary topos-theoretic approaches to constructivism; discuss a bit the contemporary approaches to constructive set theory; develop a bit more in depth the connections between constructivism and computable analysis; and give a brief overview about smooth infinitesimal analysis. Any comments, suggestions or contributions are deeply appreciated[1]
A word of warning: in exploring Brouwer's intuitionistic mathematics, we are faced with the question of how to approach his ideas; as noted by Van Stigt [vS90, pp. 296-297], Brouwer's thought evolved continuously over the years, going from a generic endorsement of the french "neo-intuitionistic" school, to a radical independence of thought. Throughout this brief note, we will for the most part not be concerned with the chronology or the structure of Brouwer's thought, seeking to privilege clarity of presentation over historical accuracy. We will also point to connections with contemporary areas of interest in mathematical logic and comment on Brouwer's overall impact on mathematical logic.

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1 Brouwer's construction of mathematics

1.1 Intuitions of Continuity

As noted by Amity in the first part of the presentation, the crucial concept that anchored Brouwer in constructing mathematics was the "Primordial Intuition" of an idealized subject [vS90, pp.172], which he used as a measuring yard to judge the validity of concepts. As such, certain concepts such as that of "set" could not be allowed to behave in the way in which people like Cantor and Zermelo thought they should, following a classical setting. In this setting, Brouwer rejected the Cantor-Dedekind Axiom:

Axiom 1.1. (Cantor-Dedekind) There is a one-to-one correspondence between the points on the continuum and some set.

Nevertheless, Brouwer seemed to note over the course of his career that approximate notions would be necessary, given the efforts he put into constructing an intuitionistic set theory [vS90, pp.297].
The first step for Brouwer in reconstructing mathematics was the following principle: **The discrete and the continuous are complementary, and both emerge from the primordial intuition** [vS90, pp.320]. For Brouwer the emergence of the naturals was rooted in the understanding of "two-ness": the contrast between successive moments in the subject's realization of time. The natural numbers arose as the process of realizing the indefinite way in which time flowed, and as such, unlike in other constructively flavoured systems of the time (like Kronecker's) Brouwer accepted "completed totalities". As we will see below, this boils down to an acceptance of discrete mathematics as essentially unproblematic.
Nevertheless, the buck stops here. The name of his movement notwihstanding, Brouwer's treatment of the cotninuum has some elements that can be quite shocking. The foremost – and the one which we will deal with here – is the so-called "Brouwer's Theorem", saying that all functions from the reals to the reals are uniformly continuous. If that is the case, What are we then to make of the following? Surely this seems

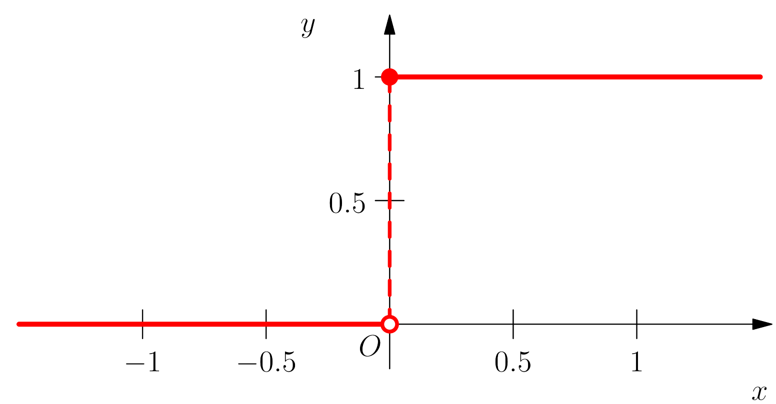

Figure 1: Heaviside "function"

enough like a function, and surely not continuous. However, one might answer as follows: if you had to define the above function, it seems like you would need some rule for doing it. The obvious solution seems to be to specify it as follows:

$$f(x) \coloneqq \begin{cases} 1 \text{ if } x \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

If we understand such a function computationally, as a procedure that transforms inputs, then the rules we wrote down gain the quality of predicates that need to be verified. So in order to know whether this is an actual function - whether it is a computationally sound process, which will terminate - we need to ask whether the conditions can be legitimately checked. Can we check whether a real number is greater than

another? The answer also seems intuitively yes, but it is important to remember that reals can only be approximated as infinite, arbitrarily set up, expansions. It is these expansions that we will analyse against 0.
So suppose we are precisely at the "point of discontinuity", call it O, and want to check in which of the two cases we are. We will begin expanding O and find that we cannot tell it apart from 0 - since indeed $O = 0$. But how can we know we are at 0? We can wait for as long as we like, but there seems to be no way to tell if O is indeed 0, or just some number which is imperciptbly close to 0, but is indeed negative.

For another example of the same vein, and to which we will also return, consider the intuitionistic proposition that says that: "*If one takes away a point from the continuum, it does not become disconnected*". Again we might ripost: what then should happen to the continuum? How are we to think about it?
But consider the notion of connectedness as not being divisible into two clopen subsets. Our intuition about taking a point away is that this point was gluing two open intervals. But how would we actually specify these open intervals? How could we make sure that all points are precisely either on one side or the other? Once again we are faced with the need to verify infinitely many possibilities, and once again this computational analogy explains the motivation for it to not become disconnected.

In both cases, and to continue the analogy with computability theory, we would have to conclude that these functions and procedures were of a "bad" kind – namely non-computable – even if we then wanted to move on to study them on those terms. For constructivists, the point is that by their very nature, we have no reason to believe that such unconstructive objects exist, and seemingly harmless objects like the Heaviside function or a point disconnecting the continuum are simply a matter of misunderstanding.
Indeed, for Brouwer, the continuum stops being a crisp place where clean breaks and cuts can be made, and becomes much more murky. Indeed, the intuitive picture we would like you (the reader) to keep in mind when reading the following is that of a syrup covered real line, such that all operations have to pass through a thick coating, and leave a trace. The classical continuum would then be the result of a thorough rinsing of the intuitionistic continuum.

1.2 From "Separable" Mathematics to the Continuum

We begin by outlining Brouwer's ideas about discrete mathematics – which he called "separable", given in his systems it could be handled by classical set theory without contradicting intuitionism. The following simply denote the usual number systems we are used to.

Definition 1.2. Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ denote the usual sets of naturals, integers and rationals, with their arithmetic properties and their order relation $<$. We denote by ω, ζ, η respectively the order types of the naturals, the integers and the rationals.

To denote concepts and propositions as seen by the lens of Brouwer's intuitionism, let **BI** denote the set of principles and conceptions which we have hinted at so far[2]. We will now try to set up a few of the key principles we will be needing.

Definition 1.3. BI is composed of the following:

- The law of excluded middle does not hold in general, but might hold in special cases.
- We work within a set theory where the constructions obey intuitionistic principles. Where not otherwise specified, "sets" refer to these objects.
- We throughout accept the schema of weak continuity (WC-N): for any relation $\phi(x, y)$ which relates choice sequences and natural numbers,

$$\forall \alpha \exists p\ \phi(\alpha, p) \rightarrow \forall \alpha \exists n \exists p \forall \beta (\beta \restriction_n = \alpha \restriction_n \rightarrow \phi(\beta, p))$$

[2] This is not supposed to be a formal system, but rather a shorthand to refer to the informal setting in which Brouwer discussed his ideas. For a more formal approach to Brouwer's mathematics, see [AST88, pp.206-]

- We accept a restricted form of the Axiom of Choice: if R is a relation on naturals, and $\forall n \exists m R(n, m)$, then there exists a function f such that $\forall n R(n, f(n))$.[3]

We will return to the principle of weak continuity laid out in the previous definitions, once we establish what choice sequences are. For now, it is important to note that for certain objects and theories, one might still have excluded middle and double negation elimination. Whenever a property P over a domain R is such that we can always determine whether $P(x)$ or $\neg P(x)$ for each $x \in R$ we say that P is **decidable**. Indeed, because of their somewhat finitary nature the canonical objects of elementary arithmetic remain unchanged:

Proposition 1.4. *For* $\mathbb{N}, \mathbb{Z}$ *and* $\mathbb{Q}$ *under* BI *agrees with the* ZFC *as concerns the arithmetic and order theory, except in the truth value of undecided propositions. In particular,* $x < y$ *is a decidable predicate for* ω, ζ *and* η.

However, one does not now proceed to "construct the continuum" in any strong sense. For Brouwer, throughout the various iterations of his thought, the continuum already existed, and was an ultimately a priori concept. For him the continuum was also tied to the conception of time, representing the phenomenological property that time was seen as a continuous and unbroken succession of events. In this sense, *analysis* was precisely the field of mathematics that sought to decompose the continuum by breaking it into small pieces and studying those pieces. To allow such an analysis to happen, then, Brouwer allowed the "construction" of points on the real line (see Figure 2), which would point to places where various continua would meet. Such points were in that sense "approximations", locations hinted at by getting closer and closer with rational numbers, and so, they would be infinite sequences, whether of rationals or of naturals. In parallel, the crucial notion of an interval in the continuum, captures the intuitive notion we speak of in temporal terms.
Combining these two notions, one may take intervals and connect them by looking at a specific point where they are glued:

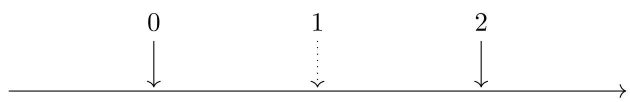

Figure 2: Constructing a point in the continuum

Following this line of thought, such points in the continuum would have to follow a law of construction, and so there would be only denumerably many. This would ultimately be abandoned by Brouwer, on practical grounds: he accepted that the continuum was important as a place to recover geometry - and so our Kantian intuition of space - which entailed that the intuitive Euclidean notions as filtered by 19th century analysis ought to remain in the continuum. But so, there should be a notion of volume as adequately represented by an idea of *measure*, which necessarily must assing the value 0 to points, and which should be σ-additive, which ties to the continuity of our notion of volume. Brouwer's conclusion is thus:

> "(...) such an ever-unfinished and ever-denumerable system of 'real numbers' is incapable of fulfilling the mathematical functions of the continuum for the simple reason that it cannot have a measure positively differing from zero" [vS90, Brouwer in VanStigt, pp.321]

A way out of this predicament is to place the rational numbers as a "grid" around the continuum, and also accept a larger continuum. This was precisely what Brouwer chose to do. The following is essentially the concept of a Cauchy sequence. We call attention to the liberal use of function and relation symbols as respects naturals and rationals, given as established before, their theory in this context is decidable.

Definition 1.5. Let $(x_n)_{n \in \omega}$ be a sequence of rational numbers. We say that $(x_n)_{n \in \omega}$ is **Cauchy** if for every k there is a p such that for every m:

$$|a_{p+m} - a_p| < 1/k$$

[3] Note that is equivalent to a form of Countable choice.

Under this conception, $a \in \mathbb{R}$ is a point if and only if there exists a sequence $(a_n)_{n\in\omega}$ "converging" to a. Given two such sequences $(a_n), (b_n)$ we define the relation $(a_n) \approx (b_n)$ as:

$$\forall k, \exists n |a_n - b_n| < 1/k$$

As a first example of how double negation elimination can still happen for intuitionistic object, we can prove that under this definition, we have double negation elimination.

Proposition 1.6. *([Dum77, Theorem 2.4]) Assume* BI*. Suppose that $a = (a_n)_{n\in\omega}$ and $b = (b_n)_{n\in\omega}$ and $a \not\approx b$ is impossible. Then $a = b$*

Proof. Fix k and let n be such that for all p:

$$|a_{n+p} - a_n| < 1/4k \text{ and } |b_{n+p} - b_n| < 1/4k$$

Suppose that $|a_n - b_n| \geq 1/k$. Then:

$$|a_{n+p} - b_{n+p}| \geq |a_{n+p} - a_n| + |a_n - b_{n+p}| \geq |a_{n+p} - a_n| + |a_n - b_n| + |b_n - b_{n+p}| \geq 1/2k$$

For all $p \in \mathbb{N}$. Hence $(a_n) \not\approx (b_n)$, which is impossible.
Thus $|a_n - b_n| < 1/k$, from which it follows that:

$$|a_{n+p} - b_{n+p}| < 2/k$$

For all $p \in \mathbb{N}$. Thus $(a_n) \approx (b_n)$. □

We can likewise define an order relation:

Definition 1.7. Let a and b be reals. We say that $a < b$ if:

$$\exists k \exists n \forall p (b_{n+p} - a_{n+p} > 1/k)$$

We write $a \not> b$ to mean $\neg(b < a)$.

As we hinted at in the previous section, the notion of being smaller than, like the notion of being equal, will not in general be decidable for real numbers. Moreover, trichotomy will not in general hold, although we will not show this here. Instead we will show that trichotomy is not decidable for arbitrary real numbers[4].

Theorem 1.8. *Assume* BI*. Then trichotomy of reals is not decidable.*

Proof. We construct (a_n) and (b_n) simultaneously. First, let T be a statement of some number theoretic result that is hard. For instance, let $T(n)$ denote the statement "Iterating the Collatz map on n eventually brings it down to 1". Now construct the two sequences as follows:

- If $\forall m \leq n$ we $T(m)$ then $a_n = b_n = 1/n$
- If $m < n$ is the least counterexample to T, and m is **odd** then $a_n = 1/m$ and $b_n = 1/n$
- If $m < n$ is as above but **even** then $a_n = 1/n$ and $b_n = 1/m$

Then notice that:

- $a = b = 0$ if and only if the Collatz Conjecture is true
- $a = 0 < b$ if and only if the least counterexample to Collatz is even

[4] For the model theoretically inclined: one might wonder how this squares with the Tarski-Seidenberg result on the decidability of the algebraic geometry of the reals. But note that semialgebraic sets tend to be defined without parameters, whilst trichotomy in the sense we give here accepts parameters.

- $a > 0 = b$ if and only if the least counterexample to Collatz is odd

Now set $c \coloneqq a - b$. We wish to decide:

$$(c < 0) \vee (c = 0) \vee (c > 0)$$

But this requires deciding whether the Collatz Conjecture is true or not. □

The role of the Collatz conjecture in the previous statement is of course absolutely arbitrary, and just makes the proof more colourful. A stronger case could be made for truly undecidable statements like the Halting problem, but we leave such considerations to the reader. It is however worthy to note that this was one of the first cases of a *weak counterexample* (for more on that see [Dum77]).

An important function which we want to be preserved - and which we will come back to later - is the absolute value function:

Proposition 1.9. *[Dum77, Theorem 2.22] Let $|\cdot|$ denote the modulus function takes the maximum of $x, -x$. Then $|x - y| \ngtr |x - z| + |z - y|$.*

Much more can be said about elementary analysis in this setting, and we point the interested reader to [Dum77, Chapter 2] for a deeper introduction to these notions. As a final comment on Brouwer's initial take on the continuum, though, we would like to note that the choice to include the rationals inside the continuum does not merely allow for the continuum to be uncountable - it forces it to be so. This is because of the peculiar nature of a well-known result:

Proposition 1.10. *Given any sequence of points in the continuum, and an interval $[a, b]$ with rational endpoints, one can find an element of $[a, b]$ not on the sequence.*

Anyone familiar with algebra or set theory will note that this is the statement of one Cantor's proofs of the uncountability of the algebraic numbers. More concretely, it tells us that assuming that there is an injection of the continuum (in the interval $[a, b]$) into the naturals leads to a contradiction. So the continuum is not countable, so long as this concept is taken to mean the existence of a bijection with the naturals (also known as "equi-potency"). As noted by Kanamori [Kan12, pp.4], despite common misconception, this is a fully constructive account, though it is not clear to us at this time whether there is any relationship between this and Brouwer's change of heart.
Nevertheless, to illustrate how this setting allows for variation, this does *not* mean that there is no surjection from a subset of the naturals to the continuum. Indeed, with the modicum of choice we are accepting, such a surjection need not split (i.e, there does not need to be an injection in the opposite direction), so it is possible for this to hold. The interested reader can find a construction of this in [Bel14, pp.79-83].

1.3 A continuum of sequences

In his later work, Brouwer eventually came to understand the role of sequences as instrumental in representing the "points" of the continuum, and in the wake of his "Second act of intuitionism", instrumental aspects of mathematical activity. In this section we will take a moment to inspect these concepts, before returning to considerations about the continuum. Our approach will be cloaked in a decidably modern language.

Definition 1.11. We denote by X^ω the (intuitionistic) set of all sequences of elements from X. We refer to elements $f \in X^\omega$ as choice sequences on X. Given such a sequence f, we say that f is **recursive**, if there exists an $n \in \omega$ and some computable function m, such that for all $k \geq n$, $f(k) = m(\langle f(k-1), ..., f(0) \rangle)$. In other words, the behaviour of f is computably determined by its history and some algorithm.
We say that f is **free-choice** if f is not computable.

As part of his secondary attempt to formalise the continuum, Brouwer argued that the existence of all free choice sequences should follow intuitionistically - in the context of his arguments, due to the inherent subjective freedom of the subject to, at any point in the construction, change his opinion about the construction. This amounts to making the following assertion:

Proposition 1.12. *There exist all free-choice sequences on ω^ω.*

As one can expect, the name “choice” is not arbitrary, although it is somewhat coincidental. To prove this proposition we can use a diagonalisation against the computable functions, and then note that such a method can effectively produce all free-choice sequences.[5]. In all cases, sequences of both cases are given as the limit of some finitary process, and it is this process that will make the way for Brouwer's distinctive proposal.

It is also here that the concept of weak continuity plays a key role. We recall that for a property holding between choice sequences and natural numbers, this says:

$$\forall\alpha\exists p\ \phi(\alpha, p) \to \forall\alpha\exists k\exists p\forall\beta(\beta\restriction_k = \alpha\restriction_k \to \phi(\beta, p))$$

What is the intuition behind this principle? We can represent it with a (stolen) picture:

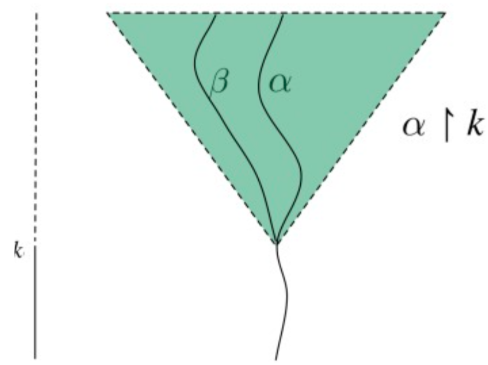

Figure 3: Representation of the weak continuity principle [Eas21]

Think of the relations ϕ as establishing properties about choice sequences. The weak continuity principle essentially says that if we were able to decide the property for every sequence, that must mean because for every sequence, a finite number of steps is all that was needed to decide that property.
Despite this harmless setup, Weak Continuity is already contradictory with the law of excluded middle. To see why, note that using the intuitionistic tautology:

$$(A \vee B) \leftrightarrow \exists x[(x = 0 \wedge A) \vee (x \neq 0 \wedge B)]$$

We can conclude:

$$\forall\alpha(A(\alpha) \vee B(\alpha)) \to \forall\alpha\exists x, y\forall\beta, \alpha\restriction_x \preceq \beta((y = 0 \wedge A(\beta)) \vee (y \neq 0 \wedge B(\beta))$$

But notice that it cannot be the case that $y = 0$ and $y \neq 0$. So we can distribute the universal quantifier, obtaining the following version of weak continuity for disjunctions (due to Troelstra and Van Dalen, see [AST88, pp.209]):

$$\forall\alpha(A(\alpha) \vee B(\alpha)) \to \forall\alpha\exists x((\forall\beta, \alpha\restriction_x \preceq \beta, A(\beta)) \vee (\forall\beta, \alpha\restriction_x \preceq \beta, B(\beta)))$$

Weak continuity is a profoundly non-classical principle. The following proposition thus serve as confirmation that the Law of Excluded Middle truly does not square with Brouwer's Intuitionism:

[5] As noted by Dummett[Dum77, pp.37] the addition of the Axiom of Choice might seem odd to accept the Axiom of Choice in a system of constructive mathematics, but this can be justified by considering the scope of quantification. Moreover, we note that this does not contradict Diaconescu's proof, since the set construction principle that would allow it to go through is similarly disallowed by Brouwer's intuitionism.

Proposition 1.13. *[AST88, Proposition 6.4] Assume* $\mathsf{BI}+\mathsf{WCN}$*. Then the Law of Excluded Middle fails.*

Proof. Assume the Law of Excluded Middle. Then consider the following instance:

$$\forall\alpha(\forall x\in\omega(\alpha(x)=0)\vee\neg\forall x\in\omega(\alpha(x)=0))$$

where α ranges over choice sequences. Then by WCN on disjunctions we have that:

$$\forall\alpha\exists k(\forall\beta,\alpha\restriction_k\preceq\beta(\forall x\in\omega\beta(x)=0)\vee\forall\beta,\alpha\restriction_k\preceq\beta(\forall x\in\omega\beta(x)\neq 0))$$

Now let α be the constant function on 0. Let k be the witness given above. Then we conclude that for any β agreeing with α up to level k:

$$\forall x(\beta(x)=0)\vee\neg\forall x(\beta(x)\neq 0)$$

But neither disjunct is true: we can take a β such that it agrees with α until k but from then on is constantly 1; and we can take $\beta=\alpha$, which refute either disjunct. □

The Weak Continuity Principle is moreover optional in constructive mathematics: many systems dispense with it, considering it too strong (like classical constructive mathematics) or too weak (preferring a strong continuity principle as below, for instance). With a computational approach we might become convinced of it: we think that the process of deciding that a property holds, or indeed, of computing a given possible value that α assumes under the relation $\phi(x,y)$, should be given by some algorithm. In that sense, the above principle tells us that such an algorithm must exist for each choice sequence. A stronger idea is to make the algorithm be uniform. We formalise these ideas in the following definitions:

Definition 1.14. Let $h: Fin(\omega)\to\omega$ be the canonical bijection. We identify natural numbers and the sequences that they code.

Definition 1.15. Let K_0 denote the class of neighbourhood functions $f:\omega^{<\omega}\to\omega$, with the following properties:

(1) For every choice sequence α, $f(\alpha\restriction_n)\neq 0$ for some $n\in\omega$;

(2) For every n and m, coding a sequence, if $f(n)\neq 0$ then $f(n)=f(n+m)$.

We call K_0^*, the subset of functions $g:\omega^\omega\to\omega$ induced by K_0, the class of neighbourhood functionals.
We now define the principle CN by saying that for any relation $\phi(x,y)$ that holds of choice sequences and natural numbers:

$$\forall\alpha\exists p\ \phi(\alpha,p)\to\exists f\in K_0\forall\alpha,\phi(\alpha,f(\alpha))$$

The notion of Strong continuity thus imposes that the way in which we come to learn that a given property holds, by virtue of an initial segment, should be given by a computationally sound process. Once again this is axiomatic, and the principles and Weak and Strong continuity can come apart.

These notions are in a sense the axiomatic foundation that underlies the principle that all functions from the reals to the reals should be continuous. They also share an interesting feature with another source of non-classical constructiveness: they restrict the space of real functions to those that are well-behaved under some notion of being well-behaved. One may compare this with Church's thesis ("All total functions are computable") or the Axiom of Determinacy ("All games on the reals are determined"), and see that indeed the structure of the continuum obtained is quite similar. Nevertheless, we will still need a bit more machinery to be able to properly explore that.

1.4 Spreads, Bars and Fans

It will not be very surprising to the set theorists that Brouwer, trained mathematically as a general topologist, would turn to combinatorial structures to investigate the continuum. For those interested in descriptive set theory, we call attention to the remarkable similarity in some of the choices of concepts made here, whether it be in what one considers the spatialisation of the continuum, or the techniques used and accepted.

Definition 1.16. Let $Fin(\omega) = (\omega^{<\omega}, \preceq)$ be the set of finite sequences of natural numbers equipped with the prefix order: given sequences $\overline{x} = \langle x_0, ..., x_n \rangle$ and $\overline{y} = \langle y_0, ..., y_k \rangle$ we say that $\overline{x} \preceq \overline{y}$ if and only for every $j \leq i$ for some fixed $i \leq k$, $x_j = y_j$.

It is clear to see that $Fin(\omega)$ is a (non-strict) partial order. We will be concerned with small subsets of these:

Definition 1.17. Let $T \subseteq Fin(\omega)$ be a partial suborder. We call this a **decidable tree** if it satisfies the following properties:

(1) $\emptyset \in T$.

(2) Whenever $\langle x_0, ..., x_n \rangle \in T$ then $\langle x_0, ..., x_i \rangle \in T$ for $i \leq n$.

(3) Whenever $\langle x_0, ..., x_n \rangle \in T$, there exists some $x_{n+1} \in \mathbb{N}$ such that $\langle x_0, ..., x_n, x_{n+1} \rangle \in T$.

(4) The property of belonging to T is decidable.

Given a decidable tree, we denote its branches by:

$$[T] \coloneqq \{f \in \omega^\omega : \forall n \in \omega, f \restriction_n \in T\}$$

We call $[T]$ a **spread**.

In other words, a spread comprises a set of sequences which are decided by an initial segment. Let us give some simple examples:

(1) The set of sequences containing only even numbers.

(2) The set of all sequences (also called the universal tree).

(3) Under ZFC, a tree inside a closed set in the baire space ω^ω is a spread.[6]

Of particular importance for our developments are *finitary* spreads:

Definition 1.18. Let T be a decidable tree. We say that $[T]$ is a finitary spread (more commonly known as a **fan**) if for each sequence $\langle x_0, ..., x_n \rangle \in T$, there are most finitely many k_i such that $\langle x_0, ..., x_n, k_i \rangle \in T$.

One might wonder why we are working with choice sequences of natural numbers rather than rational numbers. As we will note in the next chapter, this is solely due to convenience of the discussion, as the main results of this section generalise to spreads taking rational values.
The interest of fans lies in that they can be used to code certain subsets of the continuum. As such, it will be interesting to establish some properties of fans, the crucial one being a form of compactness.

Definition 1.19. Let $[T]$ be a spread, and let P be a predicate of elements of T. We say that P **bars** $[T]$ if for every $\alpha \in [T]$, there is an n such that $P(\alpha \restriction_n)$.

Definition 1.20. Let $\mathsf{FAN_D}$ denote the following principle:
Let $[T]$ be a fan and P a decidable property of finite segments which bars $[T]$. Then there is a uniform bound m such that for each $x \in [T]$ there exists $k \leq m$ and $x \restriction_k$ satisfies P. In symbols:

$$\forall n(P(n) \vee \neg P(n)) \wedge \forall \alpha \in [T], \exists n, P(\alpha \restriction_n) \rightarrow \exists k \forall \alpha \in [T], \exists z \leq k P(\alpha \restriction_z)$$

[6]This example indicates that one may alternatively define a spread as a non-empty closed set of the Baire space where all branches are non-blind.

In other words, the principle $\mathsf{FAN_D}$, which we call the decidable fan principle, tells us that for a decidable property, if this property can be found in all branches at some finite stage, at some finite stage it can be decided for all branches at once. A much stronger version of this is the following:

Definition 1.21. Let FAN denote the principle analogous to $\mathsf{FAN_D}$, except we do not require the property P to be decidable. In symbols:

$$\forall \alpha \in [T], \exists n, P(\alpha \restriction_n) \to \exists k \forall \alpha \in [T], \exists z \leq k P(\alpha \restriction_z)$$

The connection with compactness can be made precise in the following sense: let $[T]$ be any fan, and give it the initial segment topology (i.e, the topology generated by the sets of the form $O(x) = \{\alpha \in [T] : x \preceq \alpha\}$ for any $x \in T$). Let $[T] = \bigcup_{i \in I} O(x_i)$ be a cover by basic elements. We may take the index set to be ω by choice, so we can consider the following predicate:

$$P(\overline{u}) \iff \exists n, O(\overline{u}) \subseteq O(x_n)$$

Then we have that P bars $[T]$. By FAN then, there exists a k such that for every α, there is some $m \leq k$ such that $P(\alpha \restriction_k)$. But then we have a finite subcover comprised by taking:

$$\{O(x_1), ..., O(x_k)\}$$

In a similar way, $\mathsf{FAN_D}$ expresses a form of "decidable compactness": when the basis is specified on the basis of a decidable predicate, we have compactness.
Given this connection, it is not surprising that such principles are classically true: indeed, FAN is precisely the contrapositive of König's Lemma. However, it is a long known fact that König's Lemma is *not* constructively acceptable (see the discussion of Dummet [Dum77, pp.53]; this also notes that the fan principle is a distinctive feature of Brouwer's constructivism, since for instance, it is inconsistent with Church's thesis).

We now turn to showing that the fan axiom already follows from the decidable fan axiom in the presence of strong continuity.

Proposition 1.22. *[AST88, Proposition 7.4, pp.219] In* BI *we have that* $\mathsf{FAN_D} + \mathsf{CN}$ *implies* FAN.

Proof. Assume that for all $\alpha \in T$, there exists an initial segment $\alpha \restriction_n$ for which P holds. By Strong continuity, we find $f \in K_0$ such that:

$$\forall \alpha \in [T], P(\alpha_{\restriction f(\alpha)})$$

Using the relation ϕ given by: $\phi(\alpha, p)$ iff $P(\alpha \restriction_p)$ holds. By the property of being a function in K_0, let $\overline{f}$ be the function that induces f. Note that by a definable bijection, we can take $\overline{f}$ to be a function from the naturals to the naturals, by letting the naturals code finite sequences of natural numbers; as such we may write sentences like $lth(n)$ to mean "the length of the sequence coded by n". Thus consider:

$$P^*(n) := \overline{f}(n) \neq 0 \wedge lth(n) \geq \overline{f}(n) - 1$$

Because f is determined by an initial segment, for every $\alpha \in T$, $f(\alpha) = \overline{f}(\alpha \restriction_n)$ for some finite value. Moreover, because $\overline{f}(x) = p$ is decidable, the predicate P^* comes out decidable as well, and we conclude that:

$$\forall \alpha \in [T] \exists n P^*(\alpha \restriction_n)$$

In essence, P^* is the finitary approximation of the previous predicate. Thus by $\mathsf{FAN_D}$ we obtain that:

$$\exists z \forall \alpha \in [T] \exists k \leq z P^*(\alpha \restriction_k)$$

In other words:

$$\exists z \forall \alpha \in [T] \exists k \leq z \overline{f}(\alpha \restriction_k) \neq 0 \wedge k \geq \overline{f}(\alpha \restriction_k) - 1)$$

But now notice that since $\overline{f}(\alpha\restriction_k)$ already converges, then $\overline{f}(k)$ also converges and has the same value. Thus, by the original statement, we know that $P(\alpha\restriction_{\overline{f}(k)})$. So:

$$\exists z \forall \alpha \in T \exists k \leq z P(k)$$

As intended. □

As such, taking CN as a given, it will suffice to find a proof of the fan theorem. Whilst we may take this as axiomatic, we prefer to use the opportunity to introduce another interesting concept presented by Brouwer: the concept of (decidable) bar induction.

Proposition 1.23. *[Dum77, pp.54] Let P, Q be two predicates holding of finite sequences of natural numbers, and let $[T]$ be a spread. Assume that:*

- $\forall \alpha \in [T] \exists n P(\alpha\restriction_n)$ *(we say that P* ***bars*** *$[T]$);*
- *P is decidable;*
- *Every finite sequence satisfying P also satisfies Q*
- *For every finite sequence $\overline{x} = \langle x_0, ..., x_n\rangle$: if for every k, $\langle x_0, ..., x_n, k\rangle \in T$ implies that $\langle x_0, ..., x_n, k\rangle \in Q$, then $\overline{x} \in Q$ as well (we say that Q is upwards hereditary).*

Then Q holds of the empty sequence.

Given the difficult nature of the concept, it might be useful to motivate it classically.

Theorem 1.24. *Assume* ZFC*. Then bar induction holds.*

Proof. Note that in the classical setting the second condition is void. Assume that Q does not hold of the empty sequence. Contraposing the condition of upwards hereditariness, we obtain that there exists some extension of the empty sequence $\langle k\rangle$ such that $\langle k\rangle \notin Q$. By contraposing the third condition, $\langle k\rangle \notin P$ as well. We can now repeat the argument, producing an infinite sequence, which will be in the spread. But for no finite sequence will it be P, so P is not a bar. □

So we can see Bar induction as essentially guaranteeing a form of backwards induction. One important fact which we will not prove is that Bar induction already follows from Strong continuity:

Theorem 1.25. *Assume* CN*. Then bar induction is valid for any spread.*

We will instead focus on proving the principle FAN from bar induction. The following argument, due to Dummett, is intimately related to König's Lemma: indeed, it may be seen as a contrapositive version of it.

Theorem 1.26. *[Dum77, pp.54] Assume* BI*. Then* CN *proves* FAN

Proof. By Proposition 1.22 we just have to prove $\mathsf{FAN_D}$. So assume that we have a decidable predicate P which bars $[T]$. Define the following relation $R(\overline{x}, y)$ between finite sequences and natural numbers:

$$R(\overline{u}, m) \iff \forall \alpha,\ \overline{u} \preceq \alpha,\ \exists n \leq m \alpha\restriction_n \in P$$

And now we define:

$$Q := \{\overline{u} : \exists m R(\overline{u}, m)\}$$

Note that whenever $\overline{u} \in P$ then $\overline{u} \in Q$: namely, taking $m = len(\overline{u})$ we see that $R(\overline{u}, m)$ holds. Moreover, notice that given $\overline{u}$, if $\overline{u}^\frown k \in Q$ for every k which extends $\overline{u}$ in T, this means that for each extension with a single element, we can an upper bound n for each such element such that $\alpha\restriction_n \in P$. Since $[T]$ is a fan, there are only finitely many such elements, so we may take the greatest such bound, call it k. This means that for each extension of $\overline{u}$, there exists $n \leq k$ such that $\alpha\restriction_k \in P$. So we have upwards hereditariness.
By the previous theorem, CN implies bar induction. Since P and Q are in the right conditions, we obtain that Q holds of the empty sequence. Thus there exists an m such that for any sequence $\alpha \in [T]$, $\alpha\restriction_m \in P$ - which was to show. □

As a final development of these ideas, we note that applications of the continuity principle can also yield the so-called "Extended Fan Theorem". This principle which we denote by ExtFAN says that the following: if C is an extensional relation (i.e, if C respects the notion of equality) of elements of a spread $[T]$ and C bars $[T]$ then there exists a uniform bar for all extensions of a given basic sequence. In symbols:

$$\forall \alpha \in [S] \exists n,\ C(\alpha, n) \to \exists m \forall \alpha \in [S] \exists n \forall \beta, \alpha \restriction_m \preceq \beta, C(\beta, n)$$

Theorem 1.27. *[Dum77, pp.62] Assume* BI*. Then* CN *proves* ExtFAN*.*

Proof. Assume that for all $\alpha \in [T]$ we have that for some n, $C(\alpha, n)$. Using the continuity principle, let f be a function in K_0 such that for every α, $C(\alpha, f(\alpha))$. Let $g = f^{\frown +1}$, that is, we shift the arguments by 1 in the function f. Then we have that:

$$\forall \alpha \in [T], \exists m, n(g(\alpha \restriction_m) = n + 1 \text{ and } \forall r < m g(\alpha \restriction_r) = 0 \text{ and } C(\alpha, n))$$

Define an (intuitionistic) predicate R in $[T]$ as follows:

$$\overline{u} \in R \iff g(\overline{u}) > 0 \text{ and } \forall \overline{v}, \overline{v} \preceq \overline{u}, g(\overline{v}) = 0$$

In light of what we just mentioned, R bars $[T]$. So by FAN, which follows from CN, there exists an m such that:

$$\forall \alpha \in [T], \exists n \leq m \alpha \restriction_n \in R$$

So if $\alpha \in [T]$, then for some $n \leq m$ and some q we have that $g(\alpha \restriction_n) = q + 1$ and for all $r \leq n$ $g(\alpha \restriction_r) = 0$ and $C(\alpha, q)$, by definition of the function g. Now if β extends $\alpha \restriction_m$ then it extends $\alpha \restriction_n$. But then $g(\beta) = q$, since the value stabilised below. Moreover, by definition of the function g and the continuity principle, we have that $C(\beta, g(b\beta))$ holds. So we conclude:

$$\exists m \forall \alpha \in [S] \exists n \forall \beta, \alpha \restriction_m \preceq \beta C(\beta, n)$$

Which was to show. ☐

We are now finally ready to make our final strides for Continuity.

1.5 Uniform Continuity of the Reals and the shape of the Continuum

Let us look back at Section 1.2, and the definition of Cauchy sequences we gave there. These were sequences of rational numbers with a given convergence condition. Thus we may take the following axiom, which is a slight expansion of Axiom 1.12:

Axiom 1.28. There exist all free-choice sequences on $\mathbb{Q}$.

Now notice that if we look at the theory we developed in the previous sections, it did not depend in any place in the representation of reals. So we can consider the following definition:

Definition 1.29. Let $\mathbb{Q}^\omega$ be the set of choice sequences on rationals. We define $\mathbb{R} \subseteq \mathbb{Q}^\omega$ by letting $\alpha \in \mathbb{R}$ if and only if α is a Cauchy sequence.
Given a partial function $f : \mathbb{Q}^\omega \to \mathbb{Q}^\omega$ we say this function is **real-valued** if it brings Cauchy sequences to Cauchy sequences.

We have thus recovered our original notion of the continuum, with its good geometric properties, whilst still connecting it to the point-set properties we have established in the previous chapter. The notions of continuity are close to the usual ones:

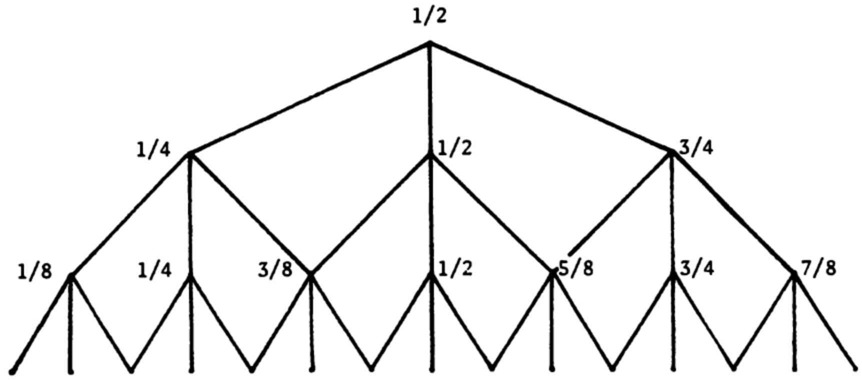

Figure 4: Ternary spread

Definition 1.30. Let f be a real-valued function, and $[u,v]$ be an interval. We say that f is continuous on $[u,v]$ at a point $a \in [u,v]$ iff:

$$\forall k \exists m \ \forall x \in [u,v], (|x-a| < 2^{-m} \to |f(x) - f(a)| < 2^{-k})$$

We say that f is continuous on $[u,v]$ if it is continuous on all $a \in [u,v]$, and that it is continuous if it is continuous on all intervals $[u,v]$.
We say that f is uniformly continuous iff:

$$\forall k \exists m \ \forall x, y \in [u,v], (|x-y| < 2^{-m} \to |f(x) - f(y)| < 2^{-k})$$

Given a k as above, we can assign it an m, by the Axiom of Choice. This function is called the *modulus of continuity*.
f is said to be uniformly continuous if it uniformly continuous in all intervals $[u,v]$.

We will now relate elements of an interval $[0,1]$ to a particular fan.

Definition 1.31. Let $S = 3^\omega$ be the ternary spread, a subset of ω^ω whose elements all assume values between 0 and 2. Given such $\alpha \in 3^\omega$ we associate it its **canonical real** $x_\alpha \in [0,1]$, understood as a choice sequence on rationals, as follows:

$$x_\alpha(i) := \big(1 + \sum_{j=0}^{i-1} \alpha(i) \cdot 2^{k-1-i}\big) \cdot 2^{-k-1}$$

Notice that the above formula describes the following fan:

The key interest in this notion is because we can indeed make the connection between rational choice sequences and this ternary spread.

Theorem 1.32. *[Dum77, pp.85] For every $y \in [0,1]$ there exists some α such that $y = x_\alpha$.*

Note that this theorem has some interesting consequences: for instance, given a real number we can find a natural number m which we can use to approximate the real number to a given precision. More specifically, if x is the given real number, and we need to approximate it to a degree of 2^{-n-1}, we can look at $x_\alpha(n+1)$, and note that by choosing the numerator to be $n+1$, we can make the difference be small enough. We will make use of this in the next theorem, which culminates our developments. The proof is taken from Dummet, pp.86.

Theorem 1.33. *(**Brouwer's Theorem**) Assume* BI+CN*. Let f be a real-valued function defined everywhere on $[0,1]$. Then f is uniformly continuous on $[0,1]$.*

Proof. Let f be everywhere defined on $[0,1]$. By the representation theorem, we can consider this as a function from $[S]$ to reals, where:

$$f(x_\alpha) = f(x) \text{ whenever } \alpha \in [S]$$

Now, for each n and $x \in [0,1]$ we can approximate $f(x)$ to within 2^{-n-1} distance, i.e:

$$\forall \alpha \in [S] \; \exists m \; |f(x_\alpha) - m \cdot 2^{-n-1}| < 2^{-n-1}$$

Note that the relation between α and m is extensional, so by the Extended Fan Theorem, for each n we find an r such that:

$$\forall \alpha \in [S] \exists m \forall \beta, \alpha \restriction_r \preceq \beta, |f(x_\beta) - m.2^{-n-1}| < 2^{-n-1}$$

So now suppose that $y, z \in [0,1]$ and $|x-y| < 2^{-r-1}$. By the representation and this fact we can choose $y = x_\alpha$ and $z = x_\beta$ such that $x_{alpha} \restriction_r = x_\beta \restriction_r$. So we have that:

$$\begin{aligned} |f(y) - f(z)| &= |f(x_\alpha) - f(x_\beta)| \\ &\not> |f(x_\alpha) - m \cdot 2^{-n-1}| + |f(x_\beta) - m \cdot 2^{-n-1}| \\ &< 2^{-n} \end{aligned}$$

Where the result follows from the intuitionistic triangle inequality and the facts we mentioned before.

□

We conclude this section by mentioning a result that refers back to the problem we noted in the beginning about connectedness of the continuum. Call a space X *decomposable* if there exist $A, B \subsetneq X$ such that $A \cup B = X$ and $A \cap B = \emptyset$. One easy consequence of Brouwer's theorem is the following:

Corollary 1.34. *The space $\mathbb{R}$ is indecomposable.*

Proof. If $\mathbb{R}$ was decomposable, write:

$$\mathbb{R} = A \cup B \text{ and } \mathbb{R} = A \cap B$$

In the intuitionistic setting, affirming that $\mathbb{R} = A \cup B$ means that we can determine for each $x \in \mathbb{R}$ which of the two sets it belongs to. So using this fact we can define a function:

$$f(x) := \begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \in B \end{cases}$$

But this function clearly cannot be uniformly continuous.□

Van Dalen further showed that this property of the continuum is indeed preserved when we take a point out, showing the thickness of the intuitionistic continuum. □

Theorem 1.35. *[vD97] Let $x \in \mathbb{R}$ be an arbitrary real. Then $\mathbb{R} - \{x\}$ is still indecomposable. Furthermore, $\mathbb{Q}^c$, the set of irrational numbers, is also indecomposable.*

One of the consequences noted by Van Dalen is that in terms of the dimension theory of the spaces, this has the consequence that the connectedness of the real continuum cannot be assembled from disparate pieces: the irrationals must already have been a thick continuum over which rational points were overlaid.

2 Brouwer's Legacy

Having proven that all functions are uniformly continuous what are we to make? In what sense was the former a *proof*? Or in a broader setting: what are we to make today of Brouwer's thought and theorems? Looking back how did this affect 20th century mathematics, and does it still have any relevance today?
The first answer to this should note that Brouwer was never constructive mathematics. His was a very peculiar strain of constructivism, which in many senses was distinct from both his and our contemporaries. Other contemporary variants of constructivism – like the constructivism of Markov, the Pre-intuitionist finitism of Kronecker, or the ultra-finitism as in Esenin-Volpin – came to wildly different conclusions, for instance defending that the continuum was countable, or that very large numbers do not exist. Contemporary constructivists – though without the philosophical baggage for the most part – can be found in the creators of smooth infinitesimal analysis, which derives the following statement, which makes Brouwer's theorem seem like a trinket:

Theorem 2.1. *Let f be a real-valued function. Then f is uniformly continuous and smooth.*

However, the historically accurate answer is that Brouwer ultimately lost. His battle with David Hilbert was met with a decisive, and astonishing defeat – in great part motivated by Brouwer's own difficulty in accepting the correctness of his derivation of principles such as bar induction, and of progressing to higher mathematics. Indeed, in 1949, Weyl – once Brouwer's most fervent supporter – wrote:

> Mathematics with Brouwer gains its highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural manner, all the time preserving the contact with intuition much more closely than had been done before. It cannot be denied, however, that in advancing to higher and more general theories the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the greater part of his towering edifice which he believed to be built of concrete blocks dissolve into mist before his eyes.

So much for reconstructing analysis. But the project was not lost. In 1967 - just a year after Brouwer died - Erret Bishop published his "Constructive Analysis", which established constructive analogues of almost all classical real analysis. This impressive feat was due to one key change in attitude: Bishop did not seek to supplant classical mathematics, but generalise it into a more grounded form of mathematics. Indeed, it is this "neutral" attitude that persisted most strongly to this day.
Brouwer's legacy – in addition to the legacy of his prime student, Heyting, who would bring intuitionistic logic to the world – is comprised of many unexpected turns, which have permeated almost all of mathematical logic. Perhaps the most striking is Brouwer's effect on Kurt Gödel, who, after reading Brouwer's ideas, became sufficiently concerned with the problems of constructibility to motivate the creation of the "constructible universe" - one of the most adeptly studied objects of set theory. Moreover, the active fields of computable analysis, reverse mathematics, owe to Brouwer many ideas and principles, including the principle FAN we proved earlier. Most importantly though, the field of Topos Theory continues to influence contemporary developments in mathematics.

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