The Computational Content of Classical Proofs

Extracting programs from classical proofs.

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Outline

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1. Friedman's A-translation
2. Program Extraction

Kreisel's theorem

Theorem (Kreisel (1958))

PA *is a conservative extension of* HA *for* Π^0_2*-sentences.*

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PA *is a conservative extension of* HA *for* Π^0_2*-sentences.*

This means that

$$\vdash_{\mathsf{PA}} \forall x \exists y \,.\, P(x,y) \iff \vdash_{\mathsf{HA}} \forall x \exists y \,.\, P(x,y),$$

where P is a computable predicate.

Corollary

A recursive function is provably total in Peano Arithmetic iff it is provably total in Heyting Arithmetic.

Some preliminaries

We first fix the language.

- $\mathcal{L}$ has logical constants $\bot, \wedge, \vee, \rightarrow, \forall, \exists$, variables $x, y, z, \ldots$, and binary predicate $=$.
- $\neg\varphi$ is an abbreviation of $\varphi \rightarrow \bot$.

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- $\mathcal{L}$ has logical constants $\bot, \wedge, \vee, \rightarrow, \forall, \exists$, variables $x, y, z, \ldots$, and binary predicate $=$.
- $\neg\varphi$ is an abbreviation of $\varphi \rightarrow \bot$.
- Terms and formulas are defined as usual.
- $\vdash_C$ resp. $\vdash_I$ denotes classical resp. intuitionistic derivability in a natural deduction system.

Double-negation translation

Definition (Gödel, Gentzen)

Let φ be a formula. Define the *double-negation translation* φ^- of φ as follows:

$$
\begin{aligned}
\bot^- &:= \bot \\
\alpha^- &:= \neg\neg\alpha, \text{ where } \alpha \neq \bot \text{ is atomic} \\
(\varphi \vee \psi)^- &:= \neg\neg(\varphi^- \vee \psi^-) \\
(\varphi \wedge \psi)^- &:= \varphi^- \wedge \psi^- \\
(\varphi \to \psi)^- &:= \varphi^- \to \psi^- \\
(\forall x.\varphi)^- &:= \forall x.\varphi^- \\
\exists x.\varphi^- &:= \neg\neg\exists x.\varphi^-
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(\varphi \lor \psi)^- &:= \neg\neg(\varphi^- \lor \psi^-) \\
(\varphi \land \psi)^- &:= \varphi^- \land \psi^- \\
(\varphi \to \psi)^- &:= \varphi^- \to \psi^- \\
(\forall x.\varphi)^- &:= \forall x.\varphi^- \\
\exists x.\varphi^- &:= \neg\neg\exists x.\varphi^-
\end{aligned}
$$

So φ^- is the result of double-negating all atomic, disjunctive and existential subformulas of φ.

Some properties of the double-negation translation

Lemma

Let φ be a formula, Γ a set of formulas, and $\Gamma^- = \{\psi^- \mid \psi \in \Gamma\}$.

1. $\vdash_C \varphi \leftrightarrow \varphi^-$,
2. $\neg\neg\varphi^- \vdash_I \varphi^-$,
3. *If $\Gamma \vdash_C \varphi$, then $\Gamma^- \vdash_I \varphi^-$ (this justifies calling it a translation),*
4. *In general not $\varphi \vdash_I \varphi^-$.*

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1, 2 and 3 are not very surprising, and their proofs are easy inductions on the depth of the derivation. 4 is less obvious. A counterexample is $\varphi = \neg\forall x.P(x)$.

Friedman's A-translation

Definition (Friedman)

Let φ and A be formulas such that no bound variable of φ is free in A. We define the *A-translation* φ^A of φ as follows:

$$\begin{aligned}
\bot^A &:= A \\
\alpha^A &:= \alpha \vee A, \text{ where } \alpha \neq \bot \text{ is atomic} \\
(\varphi \wedge \psi)^A &:= \varphi^A \wedge \psi^A \\
(\varphi \vee \psi)^A &:= \varphi^A \vee \psi^A \\
(\varphi \to \psi)^A &:= \varphi^A \to \psi^A \\
(\forall x \varphi)^A &:= \forall x \varphi^A \\
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So φ^A is the result of substituting all atomic subformulas α with $\alpha \vee A$, and replacing any $\bot$ with A.

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So φ^A is the result of substituting all atomic subformulas α with $\alpha \vee A$, and replacing any $\bot$ with A. Note that $(\neg\alpha)^A = \alpha \vee A \to A$.

Some properties of Friedman's A-translation

Lemma

Let φ be formula, Γ a set of formulas and A a formula such that φ^A and Γ^A are defined, where $\Gamma^A = \{\psi^A \mid \psi \in \Gamma\}$.

1. $\vdash_C \varphi^A \leftrightarrow \varphi \vee A$
2. $A \vdash_I \varphi^A$
3. *If* $\Gamma \vdash_I \varphi$*, then* $\Gamma^A \vdash_I \varphi^A$
4. *In general not* $\varphi \vdash_I \varphi^A$

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3. *If $\Gamma \vdash_I \varphi$, then $\Gamma^A \vdash_I \varphi^A$*
4. *In general not $\varphi \vdash_I \varphi^A$*

Proof of 1 and 2 are straight-forward inductions on the derivation. A counterexample of 4 is $\varphi := \neg\neg A$.

Sketch of proof of 3: If $\Gamma \vdash_I \varphi$, then $\Gamma^A \vdash_I \varphi^A$

The rules $\wedge_I, \wedge_E, \vee_I, \vee_E, \rightarrow_I, \rightarrow_E$ are straightforward. See for example $\rightarrow_I$:

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$\forall_I, \forall_E, \exists_I, \exists_E$ are a bit trickier because of variable bindings. We consider $\exists_I$:

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because $(\varphi[t/x])^A = \varphi^A[t/x]$ and $(\exists x.\varphi)^A = \exists x.\varphi^A$.

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because $(\varphi[t/x])^A = \varphi^A[t/x]$ and $(\exists x.\varphi)^A = \exists x.\varphi^A$.
For $\bot_E$: IH is $\Gamma^A \vdash A$, and 2 gives us $A \vdash \varphi^A$.

Arithmetic

- We add new symbols to the language:
  - nullary constant **0**,
  - unary function symbol **S**,
  - symbols $F, G, H, \ldots$ for all primitive recursive functions.

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- Peano axioms:

(refl) $x = x$

(trans) $x = y \wedge y = z \to x = z$

(cong_F) $x_i = x_i' \to F(x_1, \ldots, x_i, \ldots, x_n) = F(x_1, \ldots, x_i', \ldots, x_n)$ for any n-ary function constant F

(succ_1) $\mathbf{S}(x) \neq \mathbf{0}$

(succ_2) $\mathbf{S}(x) = \mathbf{S}(y) \to x = y$

(ind) $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(\mathbf{S}(x))) \to \forall x \varphi(x)$

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\begin{array}{rl}
(\mathit{refl}) & x = x \\
(\mathit{trans}) & x = y \wedge y = z \rightarrow x = z \\
(\mathit{cong}_F) & x_i = x_i' \rightarrow F(x_1, \ldots, x_i, \ldots, x_n) = F(x_1, \ldots, x_i', \ldots, x_n) \text{ for any } n\text{-ary} \\
 & \text{function constant } F \\
(\mathit{succ}_1) & \mathbf{S}(x) \neq \mathbf{0} \\
(\mathit{succ}_2) & \mathbf{S}(x) = \mathbf{S}(y) \rightarrow x = y \\
(\mathit{ind}) & \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(\mathbf{S}(x))) \rightarrow \forall x \varphi(x)
\end{array}
$$

$$
\begin{array}{rl}
(\mathit{proj}_F) & F(x_1, \ldots, x_i, \ldots, x_n) = x_i \\
(\mathit{comp}_F) & F(x_1, \ldots, x_n) = G(H_1(x_1, \ldots, x_n), \ldots, H_m(x_1, \ldots, x_n)) \\
(\mathit{rec}_F) & F(\mathbf{0}, x_1, \ldots, x_n) = G(x_1, \ldots, x_n) \\
 & \quad \wedge F(\mathbf{S}(y), x_1, \ldots, x_n) = H(F(y, x_1, \ldots, x_n), y, x_1, \ldots, x_n)
\end{array}
$$

Definition (Peano Arithmetic, Heyting Arithmetic)

Let Γ be a subset of the Peano axioms and φ be a formula.

- $\Gamma \vdash_C \varphi \implies \vdash_{\mathsf{PA}} \varphi$
- $\Gamma \vdash_I \varphi \implies \vdash_{\mathsf{HA}} \varphi$

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Fact

For any quantifier-free formula $\varphi(x_1, \ldots, x_n)$ there is a primitive recursive function symbol F such that

$$\vdash_{\mathsf{HA}} \varphi(x_1, \ldots, x_n) \leftrightarrow F(x_1, \ldots, x_n) = \mathbf{0}.$$

Axiom Translations

Lemma

Let φ *be a Peano axiom. Then* $\vdash_{\mathsf{HA}} \varphi^-$ *and* $\vdash_{\mathsf{HA}} \varphi^A$*.*

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If φ is on one of the forms

- α,
- $\alpha \wedge \beta$,
- $\alpha \rightarrow \beta$ or
- $\alpha \wedge \beta \rightarrow \gamma$,

where α, β, γ are atomic, then $\varphi \vdash_I \varphi^-$ and $\varphi \vdash_I \varphi^A$.

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- $\alpha \to \beta$ or
- $\alpha \wedge \beta \to \gamma$,

where α, β, γ are atomic, then $\varphi \vdash_I \varphi^-$ and $\varphi \vdash_I \varphi^A$.
Luckily, everything, except instances of the induction scheme, is of this form.

Axiom Translations

Lemma

Let φ be a Peano axiom. Then $\vdash_{\mathsf{HA}} \varphi^-$ and $\vdash_{\mathsf{HA}} \varphi^A$.

Proof.

Let φ be an instance of the induction axiom:

$$\varphi = \psi(0) \wedge \forall x(\psi(x) \to \psi(\mathbf{S}(x))) \to \forall x.\psi(x),$$

for some formula $\psi(x)$.

Axiom Translations

Lemma

Let φ be a Peano axiom. Then $\vdash_{\mathsf{HA}} \varphi^-$ and $\vdash_{\mathsf{HA}} \varphi^A$.

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Let φ be an instance of the induction axiom:

$$\varphi = \psi(0) \wedge \forall x(\psi(x) \to \psi(\mathbf{S}(x))) \to \forall x.\psi(x),$$

for some formula $\psi(x)$. Now:

$$\varphi^- = \psi^-(0) \wedge \forall x(\psi^-(x) \to \psi^-(\mathbf{S}(x))) \to \forall x.\psi^-(x),$$
$$\varphi^A = \psi^A(0) \wedge \forall x(\psi^A(x) \to \psi^A(\mathbf{S}(x))) \to \forall x.\psi^A(x),$$

which are themselves axioms of HA. □

Axiom Translations II

Corollary

1. *If* $\vdash_{\mathsf{PA}} \varphi$, *then* $\vdash_{\mathsf{HA}} \varphi^{-}$,
2. *if* $\vdash_{\mathsf{HA}} \varphi$ *and* φ^{A} *is defined, then* $\vdash_{\mathsf{HA}} \varphi^{A}$.

Axiom Translations II

Corollary

1. *If* $\vdash_{\mathsf{PA}} \varphi$, *then* $\vdash_{\mathsf{HA}} \varphi^{-}$,
2. *if* $\vdash_{\mathsf{HA}} \varphi$ *and* φ^{A} *is defined, then* $\vdash_{\mathsf{HA}} \varphi^{A}$.

Proof.

1. Let Γ be the axioms used in the derivation $\vdash_{\mathsf{PA}} \varphi$.

$$\Gamma \vdash_{C} \varphi \implies \Gamma^{-} \vdash_{I} \varphi^{-} \implies \vdash_{\mathsf{HA}} \varphi^{-}.$$

Axiom Translations II

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2. Let Γ be the axioms used in the derivation $\vdash_{\mathsf{HA}} \varphi$.

$$\Gamma \vdash_I \varphi \implies \Gamma^{A} \vdash_I \varphi^{A} \implies \vdash_{\mathsf{HA}} \varphi^{A}.$$

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Friedman's proof of Kreisel's theorem

Observation

If φ is a Σ^0_1-formula, then $\vdash_I \varphi^A \leftrightarrow \varphi \vee A$.

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- $(\exists y.F(x,y) = \mathbf{0})^A = \exists y.(F(x,y) = \mathbf{0} \vee A)$
- $\vdash_I \exists x(\varphi \vee \psi) \leftrightarrow \exists x \varphi \vee \psi$ when x not free in ψ

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- $(\exists y.F(x,y) = \mathbf{0})^A = \exists y.(F(x,y) = \mathbf{0} \vee A)$
- $\vdash_I \exists x(\varphi \vee \psi) \leftrightarrow \exists x\varphi \vee \psi$ when x not free in ψ
- Therefore $\vdash_I (\exists y.F(x,y) = \mathbf{0})^A \leftrightarrow \exists y(F(x,y) = \mathbf{0}) \vee A$

□

Friedman's proof of Kreisel's theorem

Proof of Theorem (Friedman).

- To show: $\vdash_{\mathsf{PA}} \varphi \iff \vdash_{\mathsf{HA}} \varphi$ for any Π^0_2-sentence φ.
- It is sufficient to show: $\vdash_{\mathsf{PA}} \varphi \iff \vdash_{\mathsf{HA}} \varphi$ for any Σ^0_1-formula.

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- Let $A := \exists y.F(x,y) = \mathbf{0}$.

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- Assume $\vdash_{\mathsf{PA}} A$.

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- Let $A := \exists y. F(x, y) = \mathbf{0}$.
- Assume $\vdash_{\mathsf{PA}} A$.
- Double-negation translation: $\vdash_{\mathsf{HA}} \neg\neg A$.

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- Let $A := \exists y. F(x, y) = \mathbf{0}$.
- Assume $\vdash_{\mathsf{PA}} A$.
- Double-negation translation: $\vdash_{\mathsf{HA}} \neg\neg A$.
- Friedman's A translation: $\vdash_{\mathsf{HA}} (\neg\neg A)^A$.

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- Assume $\vdash_{\mathsf{PA}} A$.
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- $\vdash_{\mathsf{HA}} (\neg\neg A)^A \leftrightarrow (((A \vee A) \to A) \to A)$

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- Double-negation translation: $\vdash_{\mathsf{HA}} \neg\neg A$.
- Friedman's A translation: $\vdash_{\mathsf{HA}} (\neg\neg A)^A$.
- $\vdash_{\mathsf{HA}} (\neg\neg A)^A \leftrightarrow (((A \vee A) \to A) \to A) \leftrightarrow A$.

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- Assume $\vdash_{\mathsf{PA}} A$.
- Double-negation translation: $\vdash_{\mathsf{HA}} \neg\neg A$.
- Friedman's A translation: $\vdash_{\mathsf{HA}} (\neg\neg A)^A$.
- $\vdash_{\mathsf{HA}} (\neg\neg A)^A \leftrightarrow (((A \vee A) \to A) \to A) \leftrightarrow A$.
- $\vdash_{\mathsf{HA}} A$.

□

Outline

1. Friedman's A-translation

2. Program Extraction

Program Extraction I

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$$\vdash t \;:\; \forall x^A\, \exists y^B \,.\, P(x,y)$$

$$\varepsilon(t) : A \to B \qquad\qquad \vdash \texttt{corr} \;:\; \forall x^A \,.\, P(x, (\varepsilon(t))(x))$$

Program Extraction II

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A perfect computer program: It does exactly what we want, and it is provably bug-free.

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Double-negation translation, A-translation:

$$\vdash_{\mathsf{PA}} t \;:\; \forall x \exists y\, P(x,y) \longrightarrow \vdash_{\mathsf{HA}} t' \;:\; \forall x \exists y\, P(x,y)$$

$$f : \mathsf{N} \to \mathsf{N} \qquad\qquad \vdash \forall x\, P(x, f(x))$$

f term in Gödel's System **T**

Extraction from Classical Proofs I

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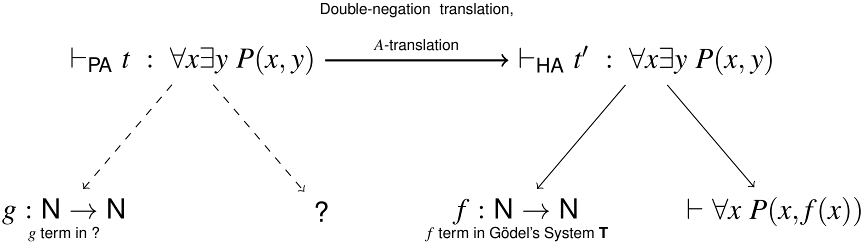

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  - Needs a more expressive programming language.
  - Griffin (1990): Classical reasoning corresponds to *control operators*.
  - Control operators allow for more flexibility; it compares to adding labels and jumps, `return` or exception handling.
- Underlying algorithms in classical proofs are potentially more efficient than ones from intuitionistic proofs.

Programs with control operators

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Extraction from Classical Proofs III

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  - CPS: Continuation Passing Style
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- Instead, we want to extract to a system that has control as a primitive construct.
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- Instead, we want to extract to a system that has control as a primitive construct.
- One approach is to interpret classical logics in a control calculus via a Curry-Howard correspondence (proofs-as-terms).
  - This requires a lot of fiddling around with reduction strategies. And program extraction tend to not necessarily be correct.
- Another approach is realisability.
  - Realisability can be seen as a formalisation of the BHK-interpretation: A realiser of an existential formula gives a witness for the formula, and a realiser of a disjunction tells which side of the disjunction is provable.

EM_1: Alwayz into somethin'

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  - $\mathsf{HA} + \mathsf{EM}_1$ proves a lot of theorems (Akama, Berardi, Hayashi, Kohlenbach 2004)
- Traditional realisability cannot be used for $\mathsf{HA} + \mathsf{EM}_1$:
- $\mathsf{HA} + \mathsf{EM}_1 \vdash \forall x \forall y (\exists z Txyz \vee \forall z \neg Txyz)$, where T is Kleene's predicate.
- A (traditional) realiser of this would solve the Halting Problem.

Learning-Based Realisability

Aschieri's Interactive Learning-Based Realisability is based on the idea of learning by counterexamples.

- Knowledge states S.
- At any state s, we have a truth value of all instances $\exists y P(x,y) \vee \forall y \neg P(x,y)$ of EM_1, and in case of $\exists y P(x,y)$ being "true", also a witness m.

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- The realiser learns:
  - At stage s: It believes $\forall x \neg P(x)$
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I will investigate whether we from $HA + EM_1$-proofs of Π^0_2-sentences can extract programs *that uses control.*

Thank you!

Counterexample to 4: In general *not* $\varphi \vdash_I \varphi^-$.

Consider a Kripke model with ω many nodes $k_0 \leq k_1 \leq k_2 \leq \ldots$, with the following domains and valuations.

i	0	1	2	$\ldots$
$D(k_i)$	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$	$\ldots$
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This proves that we cannot have $\neg\forall x.P(x) \vdash_I \neg\forall x.\neg\neg P(x)$.