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Brouwer's fixed point theorem

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Brouwer's Fixed Point Theorem

An 'almost constructive' proof

Ezra Schoen

March 31, 2021

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Brouwer's Fixed Point Theorem

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Introduction

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The Fixed Point Theorem

Theorem

Any continuous function from the closed (*n*-dimensional) disc to the closed (*n*-dimensional) disc has a fixed point; that is, a point that is left invariant by the function.

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Brouwer's fixed point theorem

Constructive variants

The Fixed Point Theorem

Theorem

Any continuous function from the closed (*n*-dimensional) disc to the closed (*n*-dimensional) disc has a fixed point; that is, a point that is left invariant by the function.

If $f: \mathbb{D}^n \to \mathbb{D}^n$ is continuous, then there is an $x \in \mathbb{D}^n$ with f(x) = x.

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A classical proof	
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A key lemma

Lemma

There is no continuous function $g : \mathbb{D}^2 \to \mathbb{S}^1$ such that f(x) = x for all $x \in \mathbb{S}^1$.

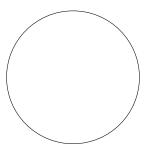
Proof: If there were such an f, then \mathbb{D}^2 and \mathbb{S}^1 would be homotopy-equivalent, but \mathbb{D}^2 is contractible, whereas $\pi_1(\mathbb{S}^1) = \mathbb{Z} \neq 0$.

Sperner's l 00000 Brouwer's fixed point theorem

Constructive variants

Constructing a function g

Assume $f : \mathbb{D}^2 \to \mathbb{D}^2$ has no fixed points.



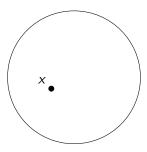
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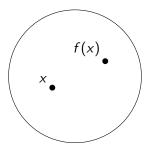
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Brouwer's fixed point theorem

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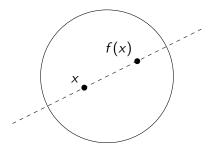
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Brouwer's fixed point theorem

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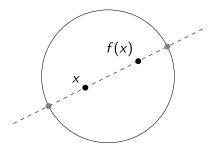
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Brouwer's fixed point theorem

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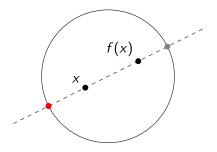
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Brouwer's fixed point theorem

Constructive variants

Constructing a function g

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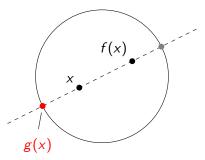
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Brouwer's fixed point theorem

Constructive variants

Constructing a function g

Assume $f : \mathbb{D}^2 \to \mathbb{D}^2$ has no fixed points.



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A classical proof	Sperner's lemma	Brouwer's fixed point theorem	Constructive variants
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Finishing the proof

If $f : \mathbb{D}^2 \to \mathbb{D}^2$ has no fixed points, then there is a continuous $g : \mathbb{D}^2 \to \mathbb{S}^1$ with g(x) = x for all $x \in \mathbb{S}^1$.

Brouwer's Fixed Point Theorem

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A classical proof	Sperner's lemma	Brouwer's fixed point theorem	Constructive variants
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Finishing the proof

If $f : \mathbb{D}^2 \to \mathbb{D}^2$ has no fixed points, then there is a continuous $g : \mathbb{D}^2 \to \mathbb{S}^1$ with g(x) = x for all $x \in \mathbb{S}^1$. Contradiction!

Unconstructivity

We can at best prove $\forall f : \neg \forall x \neg (f(x) = x)$. But this is not the same as $\forall f : \exists x (f(x) = x)!$

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A classical proof	Sperner's lemma	Brouwer's fixed point theorem	Constructive va
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The lemma

Lemma

Every Sperner coloring of a subdivision of the triangle has a rainbow triangle.

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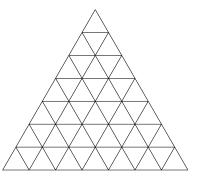
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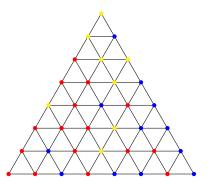
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A classical

Sperner's lemma

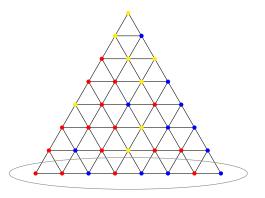
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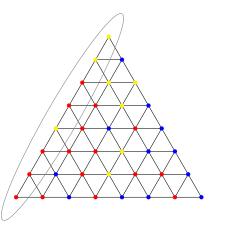
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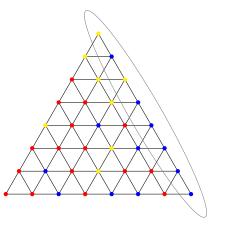
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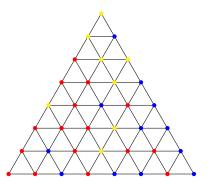
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The lemma

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Sperner's lemma

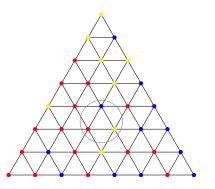
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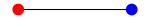
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Sperner's lemma

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Proof of Sperner's lemma (i): Doors



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Proof of Sperner's lemma (i): Doors



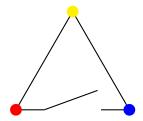
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Proof of Sperner's lemma (i): Doors



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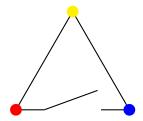
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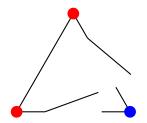
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Proof of Sperner's lemma (i): Doors





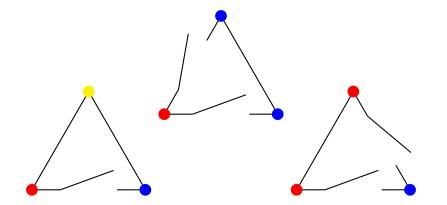
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Proof of Sperner's lemma (i): Doors



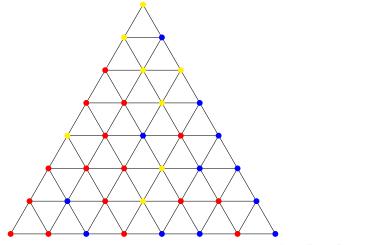
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Proof of Sperner's Lemma (ii): Walks



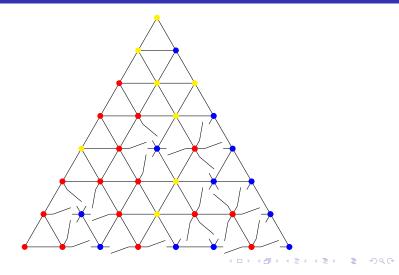
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Proof of Sperner's Lemma (ii): Walks

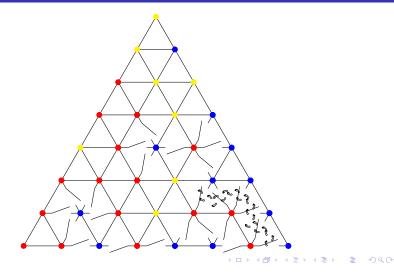


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Proof of Sperner's Lemma (ii): Walks

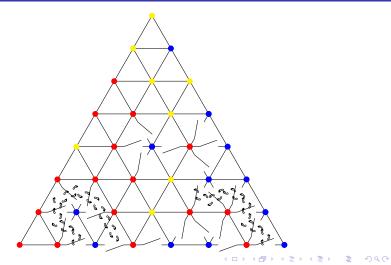


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Proof of Sperner's Lemma (ii): Walks



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Proof of Sperner's lemma (iii): Counting Doors

Every exit either leads to a rainbow triangle, or is linked to one unique other exit.

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Proof of Sperner's lemma (iii): Counting Doors

Every exit either leads to a rainbow triangle, or is linked to one unique other exit.

The number of exits is odd

Proof:

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Proof of Sperner's lemma (iii): Counting Doors

Every exit either leads to a rainbow triangle, or is linked to one unique other exit.

The number of exits is odd

Proof: Go from left to right.

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Proof of Sperner's lemma (iii): Counting Doors

Every exit either leads to a rainbow triangle, or is linked to one unique other exit.

The number of exits is odd

Proof: Go from left to right. The color changes after a segment if and only if it is an exit.

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Proof of Sperner's lemma (iii): Counting Doors

Every exit either leads to a rainbow triangle, or is linked to one unique other exit.

The number of exits is odd

Proof: Go from left to right. The color changes after a segment if and only if it is an exit. Since the start and end point have different colors, the color must change an odd number of times.

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Proof of Sperner's lemma (iii): Counting Doors

Every exit either leads to a rainbow triangle, or is linked to one unique other exit.

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So at least one exit is not linked to another,

Constructive variants

Proof of Sperner's lemma (iii): Counting Doors

Every exit either leads to a rainbow triangle, or is linked to one unique other exit.

The number of exits is odd

Proof: Go from left to right. The color changes after a segment if and only if it is an exit. Since the start and end point have different colors, the color must change an odd number of times. \Box

So at least one exit is not linked to another, hence there is a rainbow triangle.

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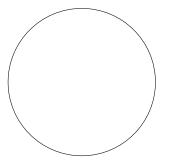
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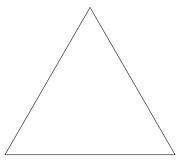
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A topological joke





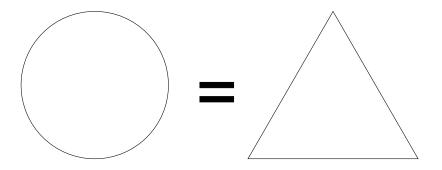
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A topological joke



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Sperner's lemma

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Coloring the standard simplex

$$\Delta^2 = \{ (t_1, t_2, t_3) \in \mathbb{R}^3 \mid t_1, t_2, t_3 \ge 0, t_1 + t_2 + t_3 = 1 \}$$

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Brouwer's Fixed Point Theorem

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Coloring the standard simplex

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Let $f: \Delta^2 \to \Delta^2$ be a function. We color t in Δ^2

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Brouwer's Fixed Point Theorem

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Let $f: \Delta^2 \to \Delta^2$ be a function. We color t in Δ^2 Red if $f(t)_1 < t_1$;

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Let $f : \Delta^2 \to \Delta^2$ be a function. We color t in Δ^2 Red if $f(t)_1 < t_1$; Blue if $f(t)_2 < t_2$;

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Coloring the standard simplex

$$\Delta^2 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid t_1, t_2, t_3 \ge 0, t_1 + t_2 + t_3 = 1\}$$

Let $f : \Delta^2 \to \Delta^2$ be a function. We color t in Δ^2 Red if $f(t)_1 < t_1$; Blue if $f(t)_2 < t_2$; Yellow if $f(t)_3 < t_3$.

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Coloring the standard simplex

$$\Delta^2 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid t_1, t_2, t_3 \ge 0, t_1 + t_2 + t_3 = 1\}$$

Let $f: \Delta^2 \to \Delta^2$ be a function. We color t in Δ^2 Red if $f(t)_1 < t_1$; Blue if $f(t)_2 < t_2$; Yellow if $f(t)_3 < t_3$.

This is a Sperner coloring!

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Coloring the standard simplex

$$\Delta^2 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid t_1, t_2, t_3 \ge 0, t_1 + t_2 + t_3 = 1\}$$

Let $f: \Delta^2 \to \Delta^2$ be a function. We color t in Δ^2

Red if $f(t)_1 < t_1$; Blue if $f(t)_2 < t_2$; Yellow if $f(t)_3 < t_3$.

This is a Sperner coloring!

Goal

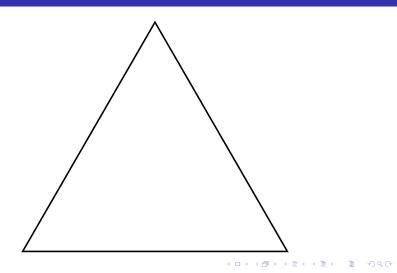
Find a shrinking sequence of rainbow triangles.

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Sperner's lemma

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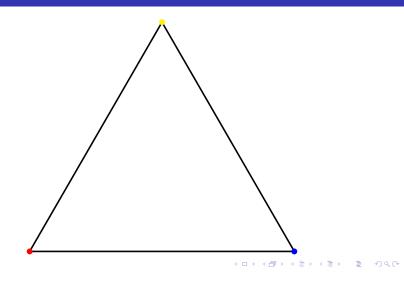
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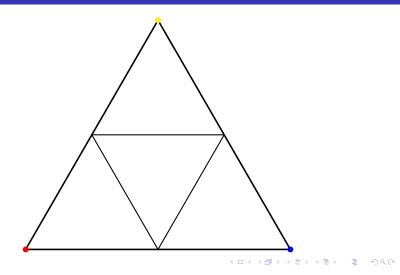


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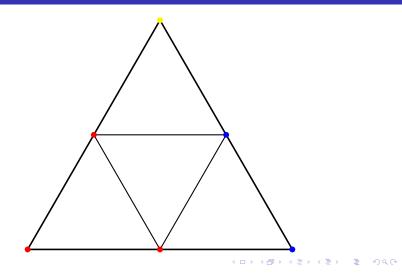


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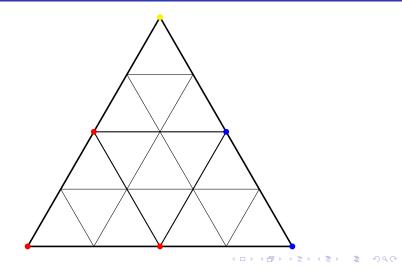
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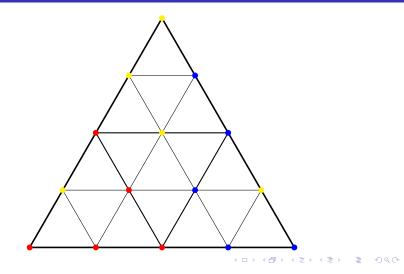


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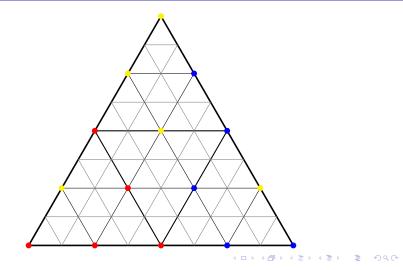
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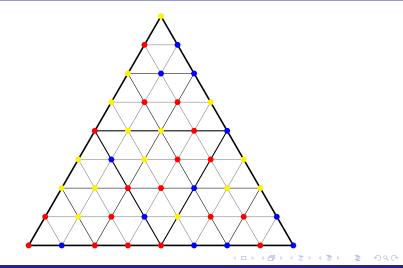


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Further subdivisions

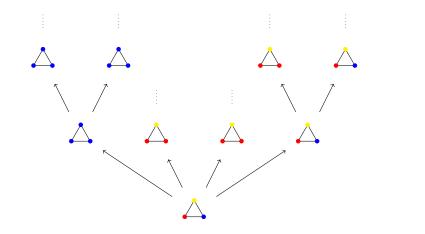


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Constructive variants

Building a tree

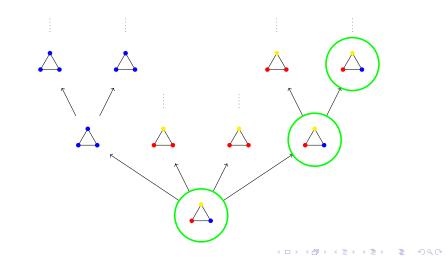


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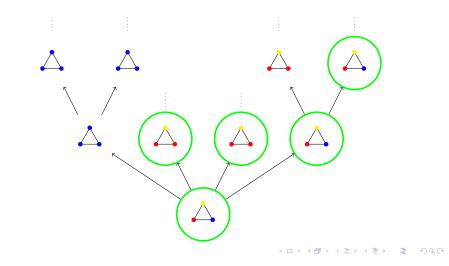
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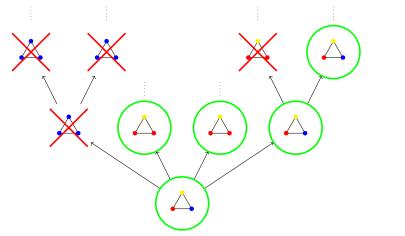
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Building a tree



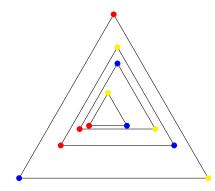
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Constructive variants 000000

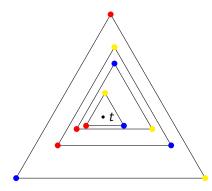
Finding the fixed point



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Finding the fixed point



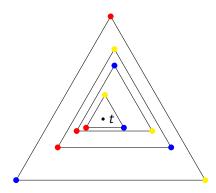
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Finding the fixed point







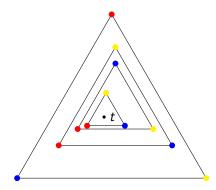
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Sperner's lemma

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Finding the fixed point



 $\bullet_i \rightarrow t \quad \Rightarrow f(t)_1 \leq t_1$ $\bullet_i \rightarrow t$ $\bullet_i \rightarrow t$

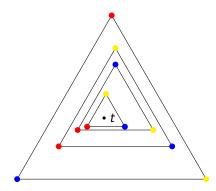
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Finding the fixed point



 $egin{aligned} \bullet_i & o t \quad \Rightarrow f(t)_1 \leq t_1 \ \bullet_i & o t \quad \Rightarrow f(t)_2 \leq t_2 \ \bullet_i & o t \end{aligned}$



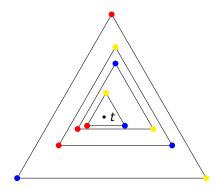
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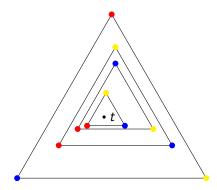
 $\begin{array}{ll} \bullet_i \to t & \Rightarrow f(t)_1 \leq t_1 \\ \bullet_i \to t & \Rightarrow f(t)_2 \leq t_2 \\ \bullet_i \to t & \Rightarrow f(t)_3 \leq t_3 \end{array}$

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 $\begin{array}{ll} \bullet_i \to t & \Rightarrow f(t)_1 \leq t_1 \\ \bullet_i \to t & \Rightarrow f(t)_2 \leq t_2 \\ \bullet_i \to t & \Rightarrow f(t)_3 \leq t_3 \end{array}$

 $t_1+t_2+t_3 = f(t)_1+f(t)_2+f(t)_3(!)$

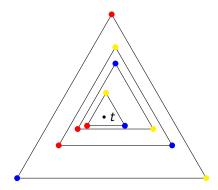
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$$t_1+t_2+t_3 = f(t)_1+f(t)_2+f(t)_3(!)$$

So $t = f(t)$.

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Introduction	A classical proof	Sperner's lemma	Brouwer's fixed point theorem	Constructive variants
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Constructive variants

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Brouwer's fixed point theorem

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Is the alternative proof constructive?

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Is the alternative proof constructive?



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Is the alternative proof constructive?



• Decidability of \leq

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Ezra Schoen Brouwer's Fixed Point Theorem

Brouwer's fixed point theorem

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Is the alternative proof constructive?

NO

• Decidability of $\leq \quad \leftarrow$ Let's not worry about this.

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Ezra Schoen Brouwer's Fixed Point Theorem

Brouwer's fixed point theorem

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Is the alternative proof constructive?

NO

- Decidability of $\leq \quad \leftarrow$ Let's not worry about this.
- Choice principle!

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Brouwer's fixed point theorem

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Is the alternative proof constructive?

NO

- Decidability of $\leq \quad \Leftarrow$ Let's not worry about this.
- Choice principle! ← Can we get rid of this?

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Is the alternative proof constructive?

NO

- Decidability of $\leq \quad \leftarrow$ Let's not worry about this.
- Choice principle! ← Can we get rid of this?

Demand more

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Brouwer's fixed point theorem

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Is the alternative proof constructive?

NO

- Decidability of $\leq \quad \leftarrow$ Let's not worry about this.
- Choice principle! ← Can we get rid of this?

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Is the alternative proof constructive?

NO

- Decidability of $\leq \quad \leftarrow$ Let's not worry about this.
- Choice principle! ← Can we get rid of this?

Demand more or Promise less

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Brouwer's fixed point theorem

Constructive variants

What choices do we make?

Weak Kőnig's lemma

If T is an infinite tree where every node has at most k children, then T has an infinite path.

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Brouwer's fixed point theorem

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What choices do we make?

Weak Kőnig's lemma

If T is an infinite tree where every node has at most k children, then T has an infinite path.

Fan theorem

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Brouwer's fixed point theorem

Constructive variants

What choices do we make?

Weak Kőnig's lemma

If T is an infinite tree where every node has at most k children, then T has an infinite path.

Fan theorem (Roughly)

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What choices do we make?

Weak Kőnig's lemma

If T is an infinite tree where every node has at most k children, then T has an infinite path.

Fan theorem (Roughly)

If T is a tree where every node has at most k children, and there are no infinite paths in T, then T has finite depth $N < \omega$.

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Brouwer's fixed point theorem

Constructive variants 000000

Demand more: Isolated fixed points

Theorem (Tanaka, 2011)

If $f : \Delta^2 \to \Delta^2$ is uniformly continuous, and every fixed point is isolated, then f has a fixed point.

Idea: If all fixed points are isolated, then in every infinite path, there is a stage after which we never have to make a choice. So by the Fan theorem, there is some *uniform bound N* on the number of necessary choices.

Brouwer's fixed point theorem

Constructive variants

Promise less: Approximate fixed points

Theorem (Van Dalen, 2009)

If $f : \Delta^2 \to \Delta^2$ is *uniformly* continuous, then for every $\epsilon > 0$ there is an x with $|x - f(x)| < \epsilon$.

Idea: In a rainbow triangle, all the corners move in different directions. But they cannot be moved very far apart; so they must stay close to their original positions.

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proof

Sperner's lemma

Brouwer's fixed point theorem

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Constructive variants

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Thank you for listening!

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