Axiomatic Social Choice

Characterisations of Voting Rules

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Thanks Ulle!

Plan for Today

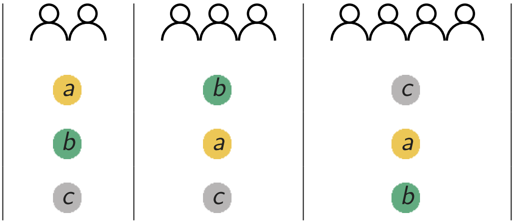
Social choice theory studies collective decision making. We will set the basics by seeing the following:

- Voting framework
- Famous voting rules
- Axiomatic characterisations
- Generalisations to incomplete inputs

All voting preliminaries can be found in the following review chapter:

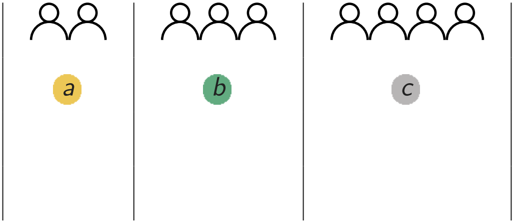
W. Zwicker. Introduction to the Theory of Voting. Handbook of Computational Social Choice, 2016.

Example: Different Rules, Different Outcomes



What should be the voting outcome?

Example: Plurality



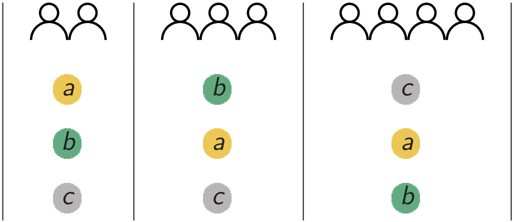
Winner with the most first positions: c

Example: Borda

			
2	*a*	*b*	*c*
1	*b*	*a*	*a*
0	*c*	*c*	*b*

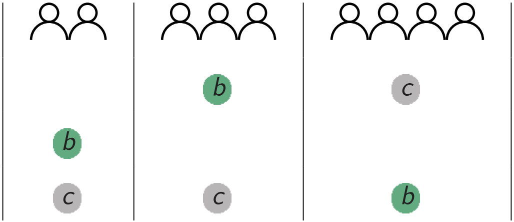
Winner with the most accumulated linear scores: *a*

Example: Plurality with Runoff—Round 1



Two alternatives with the most first positions are promoted: *b*, *c*

Example: Plurality with Runoff—Round 2



The majority alternative wins: *b*

Formal Framework

- We consider a finite set of voters $N = \{1, ..., n\}$.
- They need to choose from a finite set of m alternatives A.
- Voters have preferences and cast ballots $\succ$, which are strict linear orders over the set of alternatives $\mathcal{L}(A)$.
- All ballots of the voters together provide us with a profile:

$$\boldsymbol{P} = (\succ_1, \ldots, \succ_n) \in \mathcal{L}(A)^n$$

- A voting rule (or social choice function) selects one or more winners for each such profile:

$$F : \mathcal{L}(A)^n \to 2^A \setminus \{\emptyset\}$$

If $|F(\boldsymbol{P})| = 1$ for all profiles $\boldsymbol{P}$, then F is called resolute.

Most voting rules are irresolute. We must pair them with a tie-breaking rule (e.g., lexicographic) for a unique winner.

Positional Scoring Rules

A score vector consists of real number scoring weights:

$$\boldsymbol{w} = (w_1, \ldots, w_m), \text{ with } w_1 \geqslant \cdots \geqslant w_m \text{ and } w_1 > w_m$$

Any score vector induces a scoring rule F_w in which each voter awards w_1 points to the alternative they rank 1st, w_2 points to the 2nd-ranked, and so on. All points awarded to a given alternative are summed, and the winners are the alternatives with the greatest sum.

- Borda: $\boldsymbol{w} = (m-1, m-2, \ldots, 0)$
- Plurality: $\boldsymbol{w} = (1, 0, \ldots, 0)$
- Antiplurality (or veto): $\boldsymbol{w} = (1, \ldots, 1, 0)$
- For any $k < m$, k-approval: $\boldsymbol{w} = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$

Normative Principles and Voting Rules

Consider Plurality, Plurality with runoff, and Borda.

Which of them satisfy the following axioms?

- Anonymity: The names of the voters don't matter.
- Neutrality: The names of the alternatives don't matter.
- Monotonicity: If a winning alternative receives additional support (it is ranked higher by some voter), then it should still win the election.
- Reinforcement: If alternative a wins in two disjoint electorates, then a should also win when we join those two electorates into one.

Condorcet Principle

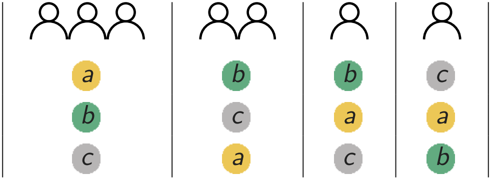
The Condorcet winner is an alternative *a* such that for every other alternative *b*, a majority of voters ranks *a* higher than *b*.

Condorcet principle: If there exists a Condorcet winner, then it should win the election. A rule satisfying this principle is a Condorcet extension.

The Borda rule infamously fails the Condorcet principle.

Positional Scoring Rules and Condorcet

Theorem: *No positional scoring rule is a Condorcet extension.*



a: $3w_1 + 2w_3 + w_2 + w_2 = 3w_1 + 2w_2 + 2w_3$

b: $3w_2 + 2w_1 + w_1 + w_3 = 3w_1 + 3w_2 + w_3$

c: $3w_3 + 2w_2 + w_3 + w_1 = w_1 + 2w_2 + 4w_3$

Because $w_1 \geqslant w_2 \geqslant w_3$, b will win, although a is the Condorcet winner.

Condorcet Extensions

Other proposals exist, often based on the majority graph of a profile: A directed graph with nodes the alternatives in A, and with an edge from a to b whenever a beats b in a pairwise majority contest.

Under the Copeland rule, an alternative gets +1 point for every pairwise majority contest won and -1 point for every such contest lost. The alternatives with the most points win. The Condorcet principle holds.

F. Brandt, M. Brill & P. Harrenstein. Tournament Solutions. Handbook of Computational Social Choice, 2016.

Other Rules and Ballots

Input rankings

- Slater: Find ranking that minimises number of edges in majority graph we'd have to switch. Elect top alternative in that ranking.
- Young: Elect alternative a that minimises the number of voters we need to remove before a becomes the Condorcet winner.

Approval Voting

- You can approve of any subset of the alternatives. The alternative with the most approvals wins.

Majority judgment

- You award a grade to each alternative ("excellent", "good", etc.). Highest median grade wins.

Axioms for Scoring Rules

Recall anonymity, neutrality, reinforcement.
Continuity says that a sufficiently large number of identical votes can always elect their first alternative.

Theorem. *A voting rule satisfies anonymity, neutrality, reinforcement, and continuity if and only if it is a scoring rule.*

P. Young. Social Choice Scoring Functions. Journal on Applied Mathematics, 1975.

Characterising the Plurality Rule

Independence of dominated alternatives (ida): If all voters rank *a* higher than *b*, then the winners should not change if we remove *b*.

Theorem. *A voting rule satisfies neutrality, anonymity, reinforcement, and ida if and only if it is the Plurality rule.*

Proof sketch.

- Lemma (by induction): If the first alternative of each voter is distinct, then all first alternatives should win.
- Then, take an arbitrary profile $\boldsymbol{P}$ and split it into sub-profiles where voters have distinct first alternatives.
- From the Lemma, all alternatives with first positions will be the winners of each sub-profile.
- By applying reinforcement repeatedly, we are left with alternatives that have the most first positions. ✓

S. Ching. A Simple Characterization of the Plurality Rule.
Journal of Economic Theory, 1996.

Characterising the Borda Rule via Condorcet

Recall that Borda fails to elect the Condorcet winner. Condorcet loser: An alternative that loses a pairwise majority contest against all others.

Theorem. *A scoring rule satisfies CL-consistency (i.e., never elects the Condorcet loser) if and only if it is the Borda rule.*

	$k\times$	$k\times$	
2	a	c	a
$1+\epsilon$	b	b	c
0	c	a	b

a: $2(k+1)$

b: $2k(1+\epsilon)$

c: $2k+1+\epsilon$

Note that if $k > \frac{1}{\epsilon}$, then b will win, although it is the Condorcet loser.

P.C. Fishburn & W.V. Gehrlein. Borda's Rule, Positional Voting, and Condorcet's Simple Majority Principle. Public Choice, 1976.

Characterising the Borda Rule via Cancellation

Cancellation: If for every two alternatives *a* and *b* the number of voters that rank *a* higher than *b* is the same as the number of voters that rank *b* higher than *a*, then all alternatives should win the election (no alternative has a clear pairwise majority advantage).

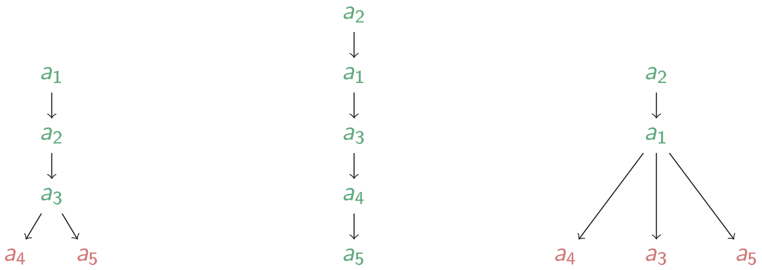
Theorem: *A scoring rule satisfies cancellation if and only if it is the Borda rule.*

P. Young. An axiomatization of Borda's rule. Journal of Economic Theory, 1974.

B. Hansson & H. Sahlquist. A proof technique for social choice with variable electorate. Journal of Economic Theory, 1976.

Truncated Orders

A truncated order strictly ranks a subset of alternatives, and places the remaining alternatives below. We define the top alternatives of a voter:



TopSet of a profile: The alternatives that are at the top for all voters.

Terzopoulou and Endriss. The Borda Class: An Axiomatic Study of the Borda Rule on Top-Truncated Preferences. Journal of Mathematical Economics, 2021.

The Borda rule on Top-Truncated Orders

Score vector in the Borda Class for k ranked alternatives:

$$(m-1, m-2, \ldots, m-k, w_{k+1}, \ldots, w_{k+1}), \text{ with } w_{k+1} < m-k$$

$m-1 \rightarrow m-2 \rightarrow \cdots \rightarrow 1 \rightarrow 0$

optimistic

$m-1 \rightarrow \cdots \rightarrow m-k \rightarrow m-k-1 \cdots m-k-1$

pessimistic

$m-1 \rightarrow \cdots \rightarrow m-k \rightarrow 0 \cdots 0$

averaged

$m-1 \rightarrow \cdots \rightarrow m-k \rightarrow \frac{m-k-1}{2} \cdots \frac{m-k-1}{2}$

Exercise: How often do these rules agree (on artificial and real data)?

Axioms for the Borda rule on Top-Truncated Orders

Theorem: *A voting rule for top-truncated orders is a scoring rule if and only if it satisfies anonymity, neutrality, reinforcement, and continuity.*

Top-cancellation: Cancellation, restricted to the TopSet of a profile.
Top-CL-consistency: Never elect the Condorcret-loser of the TopSet.

Theorem: *A scoring rule for top-truncated orders is in the Borda class if and only if it satisfies top cancellation (or top-CL consitency).*

Additional axioms characterise specific rules:

- Cancellation: The averaged Borda rule.
- Domination power (i.e., a winning alternative a can only break a tie with a different winning alternative by having its support against it strictly increased): The optimistic Borda rule.
- Bottom indifference (i.e., the number of other alternatives with which some alternative shares the bottom position does not affect its performance): The pessimistic Borda rule.

Summary

We have presented the basic voting framework of social choice, famous voting rules, and their characterisations through desirable axioms.

- Scoring rules: anonymity, neutrality, reinforcement, continuity
- Plurality: anonymity, neutrality, reinforcement, independence of dominated alternatives
- Borda: scoring rule + cancellation (or CL-consistency)

We have also discussed extensions to domains of top-truncated orders.

→ **Next**: Impossibility results.