Computing Desirable Collective Decisions I: Allocation of Indivisible Items

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Three topics:

- Monday: Allocation of indivisible items
- Tuesday: Distortion of voting rules
- Thursday: Proportional representation

Allocation of indivisible items

- $N = \{1, \ldots, n\}$ is a set of agents.
- $O = \{o_1, \ldots, o_m\}$ is a set of items/objects/goods.
- An allocation is a list $A = (A_1, \ldots, A_n)$, where $A_i \subseteq O$ is a bundle of items assigned to agent i.
  - Bundles must be pairwise disjoint.
  - We also must have $A_1 \cup \cdots \cup A_n = O$; if not, it is a partial allocation.
- Each agent i has a valuation function $v_i : 2^O \to \mathbb{R}_{\geq 0}$ that is monotonic: $B_1 \subseteq B_2 \implies v_i(B_1) \leq v_i(B_2)$. (items are goods)
- A valuation function is additive if $v_i(B) = \sum_{o \in B} v_i(\{o\})$ for all $B \subseteq O$.
  - In this case, we also write $v_i(o) := v_i(\{o\})$.
  - What are some examples of non-additive valuation functions?

Georgios Amanatidis, Haris Aziz, Georgios Birmpas, Aris Filos-Ratsikas, Bo Li, Hervé Moulin, Alexandros A. Voudouris, and Xiaowei Wu. “Fair Division of Indivisible Goods: Recent Progress and Open Questions”. In: *Artificial Intelligence* (2023), p. 103965

Example

	🏠	🧸	💎	🍭	⚽
🐕	15	3	2	2	6
🐬	7	5	5	5	7
🐦	20	3	3	3	3

Proportionality and envy-freeness

Let A be an allocation.

- A is proportional if $v_i(A_i) \geq \frac{1}{n} v_i(O)$ for every $i \in N$.
- A is envy-free if $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in N$

Question: are there examples where no envy-free allocation exists? no proportional allocation?

Proportionality and envy-freeness

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Question: are there examples where no envy-free allocation exists? no proportional allocation?

Yes. $N = \{1, 2\}$, $O = \{o_1\}$, $v_1(o_1) = v_2(o_1) = 1$.

- For the allocation $(\{o_1\}, \emptyset)$, 2 envies 1 and doesn't get proportional share.
- For the allocation $(\emptyset, \{o_1\})$, 1 envies 2 and doesn't get proportional share.

Deciding existence

Consider the following decision problem [and variant]:
EXISTENCE OF PROPORTIONAL [ENVY-FREE] ALLOCATION

- **Input**: Additive valuations $(v_i(o))_{i \in N, o \in O}$.
- **Question:** Does there exist a (complete) allocation A that is proportional? [that is envy-free?]

This problem is NP-complete.

Obvious reduction from PARTITION, works even for $n = 2$ agents.

- **Input**: List of numbers $(x_1, \ldots, x_m)$
- **Question:** Does there exist a partition (S_1, S_2) of $\{1, \ldots, m\}$ such that $\sum_{i \in S_1} x_i = \sum_{i \in S_2} x_i$?

Exercise: This only shows *weak* NP-hardness (binary encoding of numbers). Show the problem is strongly NP-hard (unrestricted n).

Some allocation rules

- Maximize utilitarian social welfare: Pick an allocation A that maximizes $\sum_{i \in N} v_i(A_i)$.
- Maximize egalitarian social welfare: Pick an allocation A that maximizes $\min_{i \in N} v_i(A_i)$.
- Maximize Nash social welfare: Pick an allocation A that maximizes $\prod_{i \in N} v_i(A_i)$.
  - This is the same as maximizing $\sum_{i \in N} \log v_i(A_i)$.
  - This is scale-free: multiplying the valuations of an agent by any factor does not change the optimal allocation.
  - It lies "between" utilitarian and egalitarian social welfare: $\min_{i \in N} v_i(A_i) \leq \sqrt[n]{\prod_{i \in N} v_i(A_i)} \leq \frac{1}{n} \sum_{i \in N} v_i(A_i)$. (AM-GM inequality)

Question: What is the computational complexity of computing optimal allocations for these objectives?

Envy-freeness up to 1 good (EF1)

An allocation is envy-free up to 1 good (EF1) if for all $i, j \in N$,

$$\text{either } v_i(A_i) \geq v_i(A_j) \text{ or there is } o \in A_j \text{ with } v_i(A_i) \geq v_i(A_j \setminus \{o\}).$$

Theorem: An EF1 allocation always exists.

Eric Budish. "The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes". In: *Journal of Political Economy* 119.6 (2011), pp. 1061–1103

Example

	🏠	🧸	💎	🍭	⚽
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Round robin rule

Consider the following procedure:

Repeatedly go through the agents in order

$$1\ 2\ 3\ \ldots n\ 1\ 2\ 3\ \ldots n\ 1\ 2\ 3\ 4$$

and on each turn, let the agent pick an unpicked good that is most valuable to them.

- Clearly, this is EF1 for agent 1 (in fact, for 1 it is envy-free).
- But it is also EF1 for everyone else. Consider for example agent 3. Let him ignore the first item that agent 1 picked, and the first item that agent 2 picked. With these ignored, no envy remains.

Question: what are some other agent orderings that guarantee EF1? what are some that don't?

Question: does this algorithm work for non-additive valuations?

Question: what are some interesting properties of this rule?

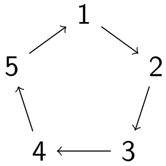
Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. "The unreasonable fairness of maximum Nash welfare". In: *ACM Transactions on Economics and Computation (TEAC)* 7.3 (2019), pp. 1–32

Envy graph, cycle elimination

Given an allocation A, its envy graph is the directed graph with 1 vertex for each agent, and an arc from i to j if i envies j.

Consider some allocation A. Suppose the envy graph has a cycle 1-2-3-4-5-1, meaning that

$$\begin{aligned}
v_1(A_1) &< v_1(A_2)\\
v_2(A_2) &< v_2(A_3)\\
v_3(A_3) &< v_3(A_4)\\
v_4(A_4) &< v_4(A_5)\\
v_5(A_5) &< v_5(A_1).
\end{aligned}$$



Then we can eliminate the cycle by giving A_2 to A_1, A_3 to A_2, etc. The resulting allocation is does not introduce any additional envy edges (and it is a Pareto improvement). If A was EF1, then same is true after.

Envy graph algorithm

1. Start with the empty (partial) allocation A.
2. While not all items are allocated:
   - Compute the envy graph for A, and update A by eliminating any cycles.
   - Now the envy graph has no cycles.
   - Pick an agent i who is a source in the envy graph, i.e. is not envied by anybody.
   - Add i's favorite unallocated item o to A_i.

Theorem: This algorithm always terminates with an EF1 allocation.

Proof: The partial allocation is EF1 throughout: Let A be allocation before adding o, and B the allocation afterwards. The only possible new EF1 violation is towards i, but

$$v_j(B_j) = v_j(A_j) \overset{i \text{ source}}{\geq} v_j(A_i) = v_j(B_i \setminus \{o\}).$$

Question: Does this algorithm work for non-additive valuations?

Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. "On approximately fair allocations of indivisible goods". In: *Proceedings of the 5th ACM Conference on Electronic Commerce (EC)*. 2004, pp. 125–131

Pareto-optimality

An allocation A is Pareto-optimal if there is no other allocation B such that $v_i(B_i) \geq v_i(A_i)$ for all $i \in N$ and $v_i(B_i) > v_i(A_i)$ for some $i \in N$.

Questions: Which rules are Pareto-optimal? Is round robin? Is envy graph?

Pareto-optimality

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Questions: Which rules are Pareto-optimal? Is round robin? Is envy graph?

Question: Does there always exist a Pareto-optimal EF1 allocation?

Maximizing Nash Welfare is PO and EF1

The MNW (Max Nash Welfare) rule selects an allocation maximizing $\prod_{i \in N} v_i(A_i)$.

Clearly, this rule is PO.*

Proved in 2016: it also satisfies EF1.

Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. "The unreasonable fairness of maximum Nash welfare". In: *ACM Transactions on Economics and Computation (TEAC)* 7.3 (2019), pp. 1–32

Nice proof due to Nisarg Shah:

- Fix any agents $i, j \in N$, and consider moving object $o \in A_j$ from A_j to A_i.
- $v_i(A_i \cup \{o\}) \cdot v_j(A_j \setminus \{o\}) \leq v_i(A_i) \cdot v_j(A_j)$.
- $\Rightarrow$: $1 - v_j(o)/v_j(A_j) \leq 1 - v_i(o)/(v_i(A_i) + v_i(o))$.
- $\Rightarrow$: $v_j(o)/v_j(A_j) \geq v_i(o)/(v_i(A_i) + v_i(o^*))$ for $o^* \in \arg\max_{o' \in A_j} v_i(o')$.
- Sum over all $o \in A_j$.

Maximizing Nash Welfare

- Used on Spliddit
- NP-hard (consider two agents with identical valuations)
- Can calculate with ILP.

> Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. "The unreasonable fairness of maximum Nash welfare". In: *ACM Transactions on Economics and Computation (TEAC)* 7.3 (2019), pp. 1–32

- `https://pref.tools/nash-indivisible/`
- There is a pseudo-polynomial algorithm achieving PO + EF1 (i.e., polynomial in n, m, $\max_{i,o} v_i(o)$).

> Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. "Finding fair and efficient allocations". In: *Proceedings of the 2018 ACM Conference on Economics and Computation (EC)*. 2018, pp. 557–574

- Nash is the unique welfarist rule (one maximizing some function of the utilities of the agents) that satisfies EF1.

> Sheung Man Yuen and Warut Suksompong. "Extending the characterization of maximum Nash welfare". In: *Economics Letters* 224 (2023), p. 111030

An algorithm for PO + EF1

Two natural strategies for designing an algorithm getting PO + EF1:

- Start with EF1 allocation, and repeatedly Pareto-improve it.
- Start with a PO allocation, and make it fairer until it is EF1.

An algorithm for PO + EF1

Fact 1: If $w_1, \ldots, w_n > 0$ are positive weights and A maximizes weighted welfare $\sum_{i \in N} w_i u_i(A_i)$, then A is Pareto optimal.

Fact 2: Allocation A maximizes weighted welfare if and only if each object o is allocated to an agent i with maximum weighted utility $w_i u_i(o)$ for o.

Fact 3: If in a weighted welfare maximizing allocation, we have for all $i, j \in N$ that $w_i u_i(A_i) \geq w_j u_j(A_j \setminus \{o\})$ for some o, then A is EF1.

High-level idea of algorithm:

- Start with $w_1 = \cdots = w_n = 1$, and let A be a utilitarian allocation.
- Try to move objects from higher-utility agents to lower-utility agents as much as possible.
- Increase weight of lowest-utility agents until they become eligible to get additional items.

Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. "Finding fair and efficient allocations". In: *Proceedings of the 2018 ACM Conference on Economics and Computation (EC)*. 2018, pp. 557–574

An algorithm for PO + EF1

The algorithm actually gets something stronger due to:

Fact 4: If $w_1, \ldots, w_n > 0$ are positive weights and A maximizes $\sum_{i \in N} w_i u_i(A_i)$, then A is fractionally Pareto optimal (fPO), i.e., is not even dominated by a fractional allocation.

- fPO is easy to check by linear programming
- when fPO is compatible with another property, there is hope for an algorithm
- but fPO is very restrictive:

	🚗 car	🎈 balloon	🧦 socks
👨 A	100	2	1
👩 B	100	1	2

dominated by

👨: 0.98 🚗 + 🎈
👩: 0.02 🚗 + 🧦

Is EF1 enough?

	car	balloon	socks
A	100	2	1
B	100	1	2

	car	balloon	socks
A	100	2	1
B	100	1	2

Envy-freeness up to any good (EFX)

Definition: An allocation A satisfies EFX if for all $i, j \in N$, and for any good $o \in A_j$, we have

$$v_i(A_i) \geq v_i(A_j \setminus \{o\})$$

- Open: Does there always exist an EFX allocation?
- Open: For positive valuation, does there always exist a PO + EFX allocation?
- Known: exists for identical valuations and for ordered valuations.
- Known: exists for two agents (easy), exists for three agents (very hard)

Bhaskar Ray Chaudhury, Jugal Garg, and Kurt Mehlhorn. "EFX exists for three agents". In: *Proceedings of the 21st ACM Conference on Economics and Computation (EC)*. 2020, pp. 1–19

Bhaskar Ray Chaudhury, Jugal Garg, Kurt Mehlhorn, Ruta Mehta, and Pranabendu Misra. "Improving EFX guarantees through rainbow cycle number". In: *Proceedings of the 22nd ACM Conference on Economics and Computation (EC)*. 2021, pp. 310–311

- Recent work about finding partial allocations that are EFX.

Ben Berger, Avi Cohen, Michal Feldman, and Amos Fiat. "Almost full EFX exists for four agents". In: *Proceedings of the AAAI Conference on Artificial Intelligence (AAAI)*. vol. 36. 5. 2022, pp. 4826–4833

EFX: Identical valuations

Suppose agents have identical valuations: $v_i(o) = v_j(o)$ for all $o \in O$ and $i, j \in N$.

If we further assume positive valuations ($v_i(o) > 0$), then a leximin allocation satisfies EFX – this is an allocation that maximizes the utility of the worst-off agent, and subject to this, maximizes the utility of the second-worst-off agent, etc.

Proof: Let A be a leximin allocation. If A fails EFX, there are i, j such that $v(A_i) < v(A_j \setminus \{o\})$. If we move o from A_j to A_i, then i becomes strictly better off but is still worse off than j. Thus the move yields an allocation that is leximin-better than A, contradiction.

Without assuming positive valuations, can use leximin++ which maximizes lowest utility, then the size of the bundle with lowest utility, then the second-lowest utility, then the size, …

Benjamin Plaut and Tim Roughgarden. "Almost envy-freeness with general valuations". In: *SIAM Journal on Discrete Mathematics* 34.2 (2020), pp. 1039–1068

- PO, works beyond additive valuations

EFX: Ordered valuations

🏠 $\succ$ 🚗 $\succ$ 💻 $\succ$ 🍭

Generalization of identical valuations: label $O = \{o_1, \ldots, o_m\}$. An instance has ordered valuations if for all $i \in N$, we have $v_i(o_1) \geq v_i(o_2) \geq \cdots \geq v_i(o_m)$.

Envy graph algorithm gives an EFX allocation for ordered valuations.

Reason (intuitively): when an object o is added during the algorithm, all agents agree that o is the worst object among allocated objects. Since the object is added to an unenvied agent i, any new envy towards i can be removed by ignoring o.

Benjamin Plaut and Tim Roughgarden. "Almost envy-freeness with general valuations". In: *SIAM Journal on Discrete Mathematics* 34.2 (2020), pp. 1039–1068

EFX: Two agents

Divide and choose

Take agent 1's valuation v_1, and consider a virtual instance with two copies of v_1, and take an EFX allocation (A_1, A_2) using one of the previous algorithms.

Let agent 2 choose their preferred of the two bundles; the other bundle goes to 1.

> Benjamin Plaut and Tim Roughgarden. "Almost envy-freeness with general valuations". In: *SIAM Journal on Discrete Mathematics* 34.2 (2020), pp. 1039–1068

EFX: Approximation

Envy graph algorithm selects an allocation that satisfies $\frac{1}{2}$-EFX.

For all $i, j \in N$: $v_i(A_i) \geq \frac{1}{2} \cdot v_i(A_j \setminus \{o\})$ for all $o \in A_j$.

Benjamin Plaut and Tim Roughgarden. "Almost envy-freeness with general valuations". In: *SIAM Journal on Discrete Mathematics* 34.2 (2020), pp. 1039–1068

Non-additive valuations?

A valuation function $v_i : 2^O \to \mathbb{R}$ is submodular if for all $A \subseteq B$ and all $x \in O \setminus B$,

$$v_i(B \cup \{o\}) - v_i(B) \leq v_i(A \cup \{o\}) - v_i(A).$$

Example: course allocation.

An EF1 allocation always exists for submodular violation.

- Open: does a PO + EF1 allocation always exist?
  Nash is not EF1.

What about chores?

A chore for agent i is an item with $v_i(o) < 0$.

We can define EF1 for mixed instances as follows:

An allocation A is EF1 if for all $i, j \in N$, either $v_i(A_i) \geq v_i(A_j)$ or there is some object $o \in A_i$ such that

$$v_i(A_i \setminus \{o\}) \geq v_i(A_j)$$

Haris Aziz, Ioannis Caragiannis, Ayumi Igarashi, and Toby Walsh. "Fair allocation of indivisible goods and chores". In: *Autonomous Agents and Multi-Agent Systems* 36 (2022), pp. 1–21

- EF1 exists via round robin, but need to be careful with envy graph.
- For 2 agents, can do PO + EF1 (adjusted winner).
- Open: can we do PO + EF1 for 3+ agents?
- Open: does EFX exist?

Things people have studied I

- Maximin fair share: similar to proportionality, but approximations exist.

> David Kurokawa, Ariel D Procaccia, and Junxing Wang. "Fair enough: Guaranteeing approximate maximin shares". In: *Journal of the ACM (JACM)* 65.2 (2018), pp. 1–27

- PROP1, PROPX, EQ1, EQX.
- Items are arranged in a line; each bundle needs to be an interval.

> Vittorio Bilò, Ioannis Caragiannis, Michele Flammini, Ayumi Igarashi, Gianpiero Monaco, Dominik Peters, Cosimo Vinci, and William S Zwicker. "Almost envy-free allocations with connected bundles". In: *Games and Economic Behavior* 131 (2022), pp. 197–221

Things people have studied II

- Best of both worlds: A lottery over EF1 allocations that is envy-free in expectation.

> Haris Aziz, Rupert Freeman, Nisarg Shah, and Rohit Vaish. "Best of both worlds: Ex ante and ex post fairness in resource allocation". In: *Operations Research* (2023)

- Weighted version: each agent deserves goods in proportion to their weight.

> Mithun Chakraborty, Ayumi Igarashi, Warut Suksompong, and Yair Zick. "Weighted envy-freeness in indivisible item allocation". In: *ACM Transactions on Economics and Computation (TEAC)* 9.3 (2021), pp. 1–39

- Some items are divisible.

> Xiaohui Bei, Zihao Li, Jinyan Liu, Shengxin Liu, and Xinhang Lu. "Fair division of mixed divisible and indivisible goods". In: *Artificial Intelligence* 293 (2021), p. 103436

- We can give agents some money to stop envy.

> Daniel Halpern and Nisarg Shah. "Fair division with subsidy". In: *Proceedings of the 12th International Symposium on Algorithmic Game Theory (SAGT)*. Springer. 2019, pp. 374–389