university of groningen

Decidability and complexity for substructural logics with weakening or contraction

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- This talk is about decidability & complexity results for substructural logics
- Substructural logics are obtained by omitting some of the structural properties of intuitionistic/classical logic...

$$\frac{X, Y, Y \Rightarrow \Pi}{X, Y \Rightarrow \Pi}\ \text{contraction} \qquad \frac{X \Rightarrow \Pi}{X, Y \Rightarrow \Pi}\ \text{weakening}$$

$$\frac{X, A, B, Y \Rightarrow \Pi}{X, B, A, Y \Rightarrow \Pi}\ \text{exchange}$$

- ...and then adding proper axioms (many, many, many possibilities)

$$(p \to q) \vee (q \to p) \qquad \neg(p \cdot q) \vee ((p \wedge q) \to p \cdot q) \qquad p^{n-1} \to p^n$$

$$\neg p \vee \neg\neg p \qquad \neg(p \cdot q)^n \vee ((p \wedge q)^{n-1} \to (p \cdot q)^n) \qquad \dots$$

  more connectives! some connectives (that can be conflated in presence of structural rules) separate in their absence. E.g. omit w or c: $\wedge$ separates as $\wedge$ and $\cdot$. Omit e: implication $\to$ separates as left and right implication.

- resource-consciousness (lots more expressivity, greater complexity)
- Many applications

software program verification	static analysis of run-time memory allocation
fuzzy systems modelling	formal reasoning about vagueness
computational linguistics	syntax and syntactic types of natural language

Theorem (R, 2020)

Every extension of FLec *that has a cut-free hypersequent calculus is decidable*

Theorem (Balasubramanian, Lang, R, 2020)

Every extension of FLew *that has a cut-free hypersequent calculus is decidable*

Theorem (Balasubramanian, Lang, R, 2020)

The above logics are in $\mathbf{F}_{\omega^\omega}$ *(hyper-Ackermannian upper bound)*

An immediate consequence:

Corollary

The fuzzy logic $\mathsf{MTL} = \mathsf{FLew} + (p \to q) \vee (q \to p)$ *is in* $\mathbf{F}_{\omega^\omega}$

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- FLec $\approx$ intuitionistic sequent calculus omitting weakening
  FLew $\approx$ intuitionistic sequent calculus omitting contraction
- decidability problem: F provable in FLec + Ax? ($F \in$ FLec + Ax?)
- Hypersequent calculi extend sequent calculi (multisets of sequents) and support cut-free proof systems for many substructural logics

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- Algebraic semantics as subvarieties of FL-algebras

  $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \backslash, /, 1, 0\rangle$ is *FL-algebra* if $\langle A, \vee, \wedge, \cdot, \backslash, /, 1\rangle$ is a *residuated lattice* and $0 \in A$
- A FL-algebra satisfying

  $x \cdot y \leq y \cdot x$ ($\forall x, y \in A$) is called *commutative*

  $0 \leq x \leq 1$ ($\forall x \in A$) is called *weakenable* (integral & zero-bounded)

  $x \leq x \cdot x$ ($\forall x \in A$) is called *contractive*
- FLec = logic of commutative contractive FL-algebras
  FLew = logic of commutative weakenable FL-algebras

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The above logics are in $\mathbf{F}_{\omega^\omega}$ *(hyper-Ackermannian upper bound)*

- $\mathbf{F}_{\omega}$ = decision problems whose running time is primitive recursive functions composed with single Ackermannian function

  $\mathbf{F}_{\omega^\omega}$ = decision problems whose running time is multiply-recursive functions composed with single hyper-Ackermannian function

- Urquhart 1999 showed that FLec is in $\mathbf{F}_{\omega}$ with matching lower bound

This talk is joint work and based on the following.

1. Extended Kripke lemma and decidability for hypersequent substructural logics. RR. LICS 2020.
2. Decidability and Complexity in Weakening and Contraction Hypersequent Substructural Logics.
   A. R. Balasubramanian, Timo Lang, RR. LICS 2021.

A. R. Balasubramanian
TU Munich

Timo Lang
UCL

$\hookrightarrow$ starting point Kripke and Urquhart

Kripke's proof of decidability applied to FLec (1959)

Multiplicative fragment

$$\frac{}{p \Rightarrow p} \qquad \frac{X, Y, Y \Rightarrow C}{X, Y \Rightarrow C}\ \text{contraction}$$

$$\frac{A, B, X \Rightarrow C}{A \cdot B, X \Rightarrow C} \qquad \frac{X \Rightarrow A \qquad Y \Rightarrow B}{X, Y \Rightarrow A \cdot B}$$

$$\frac{A, X \Rightarrow B}{X \Rightarrow A \to B} \qquad \frac{X \Rightarrow A \qquad B, Y \Rightarrow C}{A \to B, X, Y \Rightarrow C}$$

$$\frac{}{\Rightarrow 1} \qquad \frac{X \Rightarrow C}{1, X \Rightarrow C} \qquad \frac{X \Rightarrow}{X \Rightarrow 0} \qquad \frac{}{0 \Rightarrow}$$

Additive rules

$$\frac{A_i, X \Rightarrow C}{A_1 \wedge A_2, X \Rightarrow C} \qquad \frac{X \Rightarrow A \qquad X \Rightarrow B}{X \Rightarrow A \wedge B}$$

$$\frac{A, X \Rightarrow C \qquad B, X \Rightarrow C}{A \vee B, X \Rightarrow C} \qquad \frac{X \Rightarrow A_1}{X \Rightarrow A_1 \vee A_2}$$

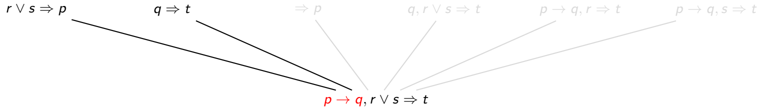
No cut-rule!

$\hookrightarrow$ backward proof search tree

Checking provability in FLec via backward proof search

1. INPUT: formula OUTPUT: YES (it is provable) / NO

2. Backward proof search all possible premises as children of conclusion



Only subformulas of input occur: cut-elimination↪subformula property

3. if it terminates then we obtain decision procedure

F is provable iff subtree of proof search tree is a proof

4. No termination since contraction can be applied backwards indefinitely

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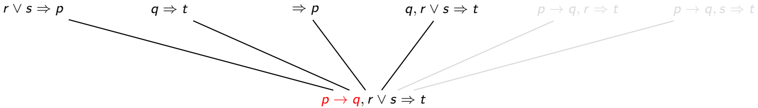

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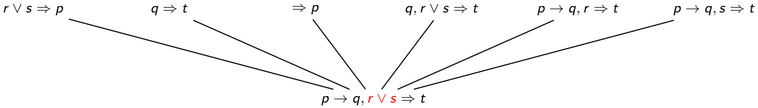

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↪ no termination since contraction

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$$\vdots$$
$$A, A, A \Rightarrow B$$
$$A, A \Rightarrow B$$
$$A \Rightarrow B$$

↪ how to solve contraction/get termination?

Structural proof theory: if there's a problematic rule. . .

$\hookrightarrow$ absorbing contraction

Structural proof theory: if there's a problematic rule... eliminate it!

$\hookrightarrow$ absorbing contraction

IDEA: permute contraction rules upwards as much as possible

$$\dfrac{\dfrac{p, r, r \Rightarrow q}{r, r \Rightarrow p \to q}\,{\to}\mathrm{R}}{r \Rightarrow p \to q}\,\mathrm{c} \quad \rightsquigarrow \quad \dfrac{\dfrac{p, r, r \Rightarrow q}{p, r \Rightarrow q}\,\mathrm{c}}{r \Rightarrow p \to q}\,{\to}\mathrm{R}$$

When permutation is impossible. . .

$$\dfrac{p \to q \Rightarrow p \qquad p \to q, q \Rightarrow}{\underbrace{p \to q, p \to q, p \to q}_{3} \Rightarrow}\,{\to}\mathrm{L}$$

e.g. no way to permute c above this $\to$L

Absorb c instead of permuting it (i.e. add following variant rules to calculus)

$$\dfrac{p \to q \Rightarrow p \qquad p \to q, q \Rightarrow}{\underbrace{p \to q, p \to q}_{2} \Rightarrow}\,{\to}\mathrm{L}^1$$

variant: one implicit contraction

$$\dfrac{p \to q \Rightarrow p \qquad p \to q, q \Rightarrow}{p \to q \Rightarrow}\,{\to}\mathrm{L}^2$$

variant: two implicit contractions

Curry's lemma

We obtain a new calculus by adding the finitely many variant rules.

Lemma (Curry's lemma: hp contraction in new calculus)
If $X, Y, Y \Rightarrow C$ provable then $X, Y \Rightarrow C$ provable with no greater height

We are not yet home since variant rules incorporate some amount of contraction. . .

. . . so sequents can get bigger upwards

$$\frac{A \to B \Rightarrow A \qquad A \to B, B \Rightarrow}{A \to B \Rightarrow} \to \mathsf{L}^2$$

If $|A| \gg |B|$ then the left premise is much bigger than the conclusion

$\hookrightarrow$ representing sequents

Aside. Simplifying the notation: LHS as sequent as n-tuple

Represent the LHS of sequent in a proof of F as element in $\mathbb{N}^{|\mathsf{subf}(F)|}$:

Fix an ordering of the subformulas of F

$$\text{Suppose } \mathsf{subf}(F) = \{p, q, r, r \to q\}$$

$$q, r \to q, q, p \Rightarrow r \quad \text{written as} \quad (\overset{p}{1}, \overset{q}{2}, \overset{r}{0}, \overset{r \to q}{1}) \Rightarrow r$$

$\hookrightarrow$ define a branch termination condition

Terminating proof search tree via redundancy

ver 1 (repetition check). A repetition of a sequent on the branch is detected

$u_{N+1} = (2,2) \Rightarrow C$

u_N

$\ldots$

$u_3 = (2,2) \Rightarrow C$

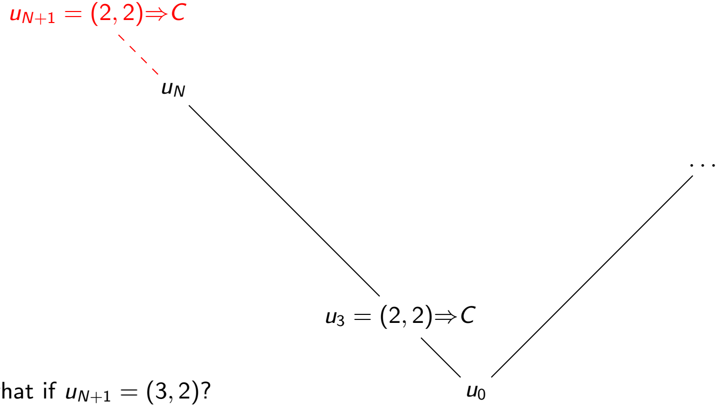
using 2-tuples just for this example

in general it is $|\mathrm{subf}(F)|$-tuples;

u_0

Terminating proof search tree via redundancy

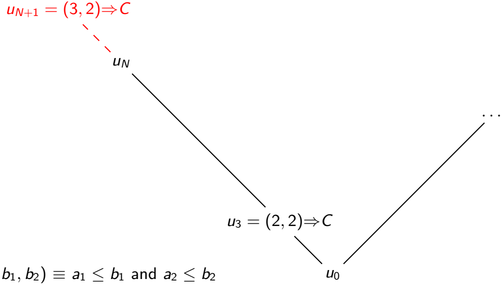
ver 1 (repetition check). $u_{N+1} = u_3$ hence u_{N+1} is redundant: any proof above it can be planted at u_3



what if $u_{N+1} = (3,2)$?

Terminating proof search tree via redundancy

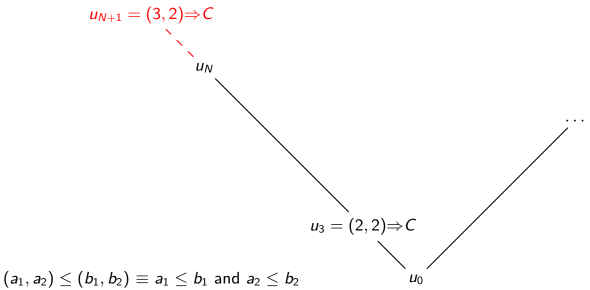
ver 2 (order check). $u_{N+1} \geq u_3$ hence u_{N+1} is redundant: any proof above it can be made into a proof of u_3 of no greater height via Curry's lemma



$(a_1, a_2) \leq (b_1, b_2) \equiv a_1 \leq b_1$ and $a_2 \leq b_2$

↪ enough to get finite proof search tree

Terminating proof search tree via redundancy



1. every branch $(u_0, u_1, \ldots, u_N)$ is a bad sequence i.e. $i < j$ implies $u_i \not\leq u_j$

   bad sequences example 1: $5, 2, 1, 0$

   example 2: $(2, 2), (1, 1), (0, 3), (0, 2), (0, 1), (0, 0)$

2. $(\mathbb{N}^k, \leq)$ is a well-quasi-ordering i.e. every infinite sequence has an increasing pair $u_i \leq u_j$ with $i < j$
   i.e. an infinite sequence cannot be a bad sequence
3. finitely branching tree & no infinite branch = proof search tree finite

$\hookrightarrow$ complexity

Urquhart's tight complexity bounds (1999)

1. upper bound: what is the height of the proof-search tree under redundancy check?
   This is the dominant term for complexity
2. No bound in general for bad sequences. After all:
3. $(1, 0), (0, 100)$ or even $(1, 0), (0, 1000)$ ... i.e. arbitrarily large jumps
4. However no rule in FLec witnesses such a great jump from conclusion to premise

   (for fixed calculus: every premise is some fixed polynomial in size of conclusion)

$\hookrightarrow$ controlled bad sequences

Controlled bad sequences

(Figueira, Figueira, Schmitz, Schnoebelen, 2011) (Schmitz, Schnoebelen, 2011)

1. bad sequence $a_0, a_1, \ldots$ is (g, n)-controlled over a normed wqo $(A, \| \|, \leq_A)$ if there is a primitive recursive g s.t.

   $$\|a_0\| \leq n \qquad \|a_1\| \leq g(n) \qquad \|a_2\| \leq g(g(n)) \qquad \|a_k\| \leq g^k(n)$$

   and $\{a \in A \text{ s.t. } \|a\| \leq n\}$ finite for every $n \in \mathbb{N}$

2. dominant term in complexity: max length of bad sequence

3. The length function theorem expresses this length. Since

4. $\exists$ control function g bounding premise size in terms of conclusion

5. FLec decision problem is in $\mathbf{F}_\omega$ i.e. primitive recursive functions composed with a single application of an Ackermannian function

6. Urquhart showed that this is tight by giving matching lower bounds.

7. Also: implicational fragment is 2*EXPTIME*-complete (Schmitz, 2016).

$\hookrightarrow$ questions? extending to other logics

Extending Kripke's argument to more logics

1. "Meyer had a bit of a problem at this point. He knew that the conclusion was true... but he did not believe it. Visions of [infinite irredundant sequences] fluttered through his dreams... he wanted an argument that he did believe" (Riche and Meyer, 1998)... Dickson's lemma

2. Kripke's decidability argument is not too sensitive to the form of the proof rules

3. subformula property, contraction absorption, and suitable wqo to get finiteness of irredundant proof trees

4. How can we extend to other logics? Some isolated results since 1959

5. sequent calculus meta-language too restrictive for cut-elimination/subformula property

   Solution: extend meta-language to get cut-freeness

   In other words: use a different type of proof system where cut-elimination holds

$\hookrightarrow$ hypersequents

Hypersequent calculus - a calculus on multisets of sequents

E.g. of a hypersequent $p, q \Rightarrow r \mid p \wedge q \Rightarrow \mid r \Rightarrow r \vee p$

Example of a hypersequent rule

$$\frac{\cdots \mid \cdots \mid \cdots \mid X_1, X_2 \Rightarrow B \qquad \cdots \mid \cdots \mid \cdots \mid Y_1, Y_2 \Rightarrow C}{\cdots \mid \cdots \mid \cdots \mid X_1, Y_1 \Rightarrow B \mid Y_2, Y_2 \Rightarrow C} \text{com}$$

Hypersequent calculi invented independently (Mints, Pottinger, Avron)

Let HFLe denote hypersequent calculus for FLe

1. lots of extensions of FLe have cut-free hypersequent calculi (Ciabattoni Galatos Terui 2008)
2. above paper: lots of extensions of FLec and FLew have cut-free hypersequent calculi. Our results will apply to all these calculi
3. Independent characterisation of extensions via substructural hierarchy

$\hookrightarrow$ representing hypersequents

Representing hypersequents in $(\mathcal{P}_f(\mathbb{N}^n))^{n+1}$

1. Let F_0 be empty formula
2. A hypersequent built from formulas $F_1, \ldots, F_n$ is written

$$\overbrace{X_1 \Rightarrow F_0}^{\text{sequent also called component}} \mid X_2 \Rightarrow F_0 \mid \ldots \mid X_{k_0} \Rightarrow F_0 \mid$$
$$Y_1 \Rightarrow F_1 \mid Y_2 \Rightarrow F_1 \mid \ldots \mid Y_{k_1} \Rightarrow F_1 \mid$$
$$\ldots$$
$$Z_1 \Rightarrow F_n \mid Z_2 \Rightarrow F_n \mid \ldots \mid Z_{k_n} \Rightarrow F_n$$

$\hookrightarrow$ hypersequent is an element of

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$$\overbrace{X_1 \Rightarrow F_0 \mid X_2 \Rightarrow F_0 \mid \ldots \mid X_{k_0} \Rightarrow F_0}^{\{X_1,\ldots,X_{k_0}\} \in \mathcal{P}_f(\mathbb{N}^n)} \mid$$
$$Y_1 \Rightarrow F_1 \mid Y_2 \Rightarrow F_1 \mid \ldots \mid Y_{k_1} \Rightarrow F_1 \mid$$
$$\ldots$$
$$Z_1 \Rightarrow F_n \mid Z_2 \Rightarrow F_n \mid \ldots \mid Z_{k_n} \Rightarrow F_n$$

3. So a hypersequent is an element of

$$\underbrace{\mathcal{P}_f(\mathbb{N}^n) \times \mathcal{P}_f(\mathbb{N}^n) \times \ldots \times \mathcal{P}_f(\mathbb{N}^n)}_{n+1}$$

$\hookrightarrow$ what else to get FLec extensions

HFLec extensions: what do we need to extend?

1. absorb contraction by adding variant rules

$$\frac{h_1 \qquad h_N}{h_0}\ r \qquad\qquad \text{original}$$

$$\frac{h_1 \qquad h_N}{g}\ r^{(k,l)} \text{ with } h_0 \rightsquigarrow_c^k h' \rightsquigarrow_{EC}^l g \qquad\qquad \text{variants } k \leq K, l \leq L$$

2. need to show that these variants suffice to eliminate all contractions

3. For $(X_1, \ldots, X_{n+1}), (Y_1, \ldots, Y_{n+1}) \in (P_f(\mathbb{N}^n))^{n+1}$ define

$$(X_1, \ldots, X_{n+1}) \leq_{\min} (Y_1, \ldots, Y_{n+1}) \text{ iff } \forall y \in Y_i \exists x \in X_i (x \leq y) \text{ for every } i$$

4. $\mathbf{X} \leq_{\min} \mathbf{Y}$ means we can go from $\mathbf{Y}$ to $\mathbf{X}$ by hypersequent Curry's lemma

5. Using length function theorem for controlled bad sequences for this wqo (Balasubramanian, 2020): decision problem for each of the FLec extensions under consideration is in $\mathbf{F}_{\omega^\omega}$

6. single application hyper-Ackermannian & multiply-recursive functions

↪ summary

What we have seen so far

INPUT: formula of size n (so at most n subformulas)

branch of naive proof search tree	$\leftrightsquigarrow$	sequence in $(\mathcal{P}_f(\mathbb{N}^n))^{n+1}$
branch with no hypersequent hp-contractible to an earlier hypersequent (order check)	$\leftrightsquigarrow$	bad sequence in $(\mathcal{P}_f(\mathbb{N}^n))^{n+1}$
no infinite bad sequence (wqo)	$\leftrightsquigarrow$	proof search terminates
proof search with order check & premise size is fixed polynomial in conclusion	$\leftrightsquigarrow$	a branch is a controlled bad sequence in $(\mathcal{P}_f(\mathbb{N}^n))^{n+1}$
if there is length function theorem: max length for controlled bad sequences	$\leftrightsquigarrow$	there is a max length for a branch $\hookrightarrow$ upper bound

$\hookrightarrow$ case of weakening

Extensions of HFLew: contraction replaced by weakening

$$\frac{X \Rightarrow A}{X, Y \Rightarrow A} \text{ weakening}$$

1. Prominent logic: monoidal t-norm based fuzzy logic

   $$\text{MTL} = \text{FLew} + (p \to q) \vee (q \to p) \text{ prelinearity axiom}$$

   Describes the common behaviours of *all* fuzzy logics based on left-continuous t-norms

2. Previous argument insufficient when c replaced by w
   If we encounter $(4, 4)$ we can prohibit smaller elements like $(4, 3)\dots$

   $$\frac{(4, 3) \Rightarrow F}{(4, 4) \Rightarrow F} \text{ height-preserving weakening}$$

   But how to prohibit infinitely many larger elements? (infinite branch)

   $$(4, 4), (4, 5), (4, 6), \dots, (4, 100), \dots$$

3. Time to go down the Lambek calculus | forward proof search |

↪ forward proof search

Forward proof search from input F

S_0 is the (finite) set of initial sequents built from subformulas in F

Def of S_1

$u \in S_0$

$\vdots$

weakening

(how much?)

$\vdots$

$$\frac{v}{w}\ r$$

$?w \in S_1?$

$u \in S_0$

$\vdots$

essential

weakening

$\vdots$

surplus

weakening

$$\frac{v}{w}\ r$$

$\rightsquigarrow$

$u \in S_0$

$\vdots$

essential

weakening

$\vdots$

$$\frac{\exists v'}{w' \in S_1}\ r$$

surplus

weakening

$w \notin S_1$

1. surplus weakening: weakening that is permutable from before r to after r
2. Obtain $(S_0, S_1, \ldots)$ s.t. S_{i+1} finite and computable from S_i
3. what 'essential" means depends on the rules in the calculus
4. aim: show there exists N s.t. $S_{N+1} = S_N$

$\hookrightarrow$ what else to get FLew extensions

What do we need to extend?

1. a hypersequent is an element of $(\mathcal{P}_f(\mathbb{N}^n))^{n+1}$
2. For $(X_1, \ldots, X_d), (Y_1, \ldots, Y_d) \in (P_f(\mathbb{N}^n))^{n+1}$ define

   $(X_1, \ldots, X_{n+1}) \leq_{\text{maj}} (Y_1, \ldots, Y_{n+1})$ iff $\forall x \in X_i \exists y \in Y_i (x \leq y)$ for every i
3. $\mathbf{X} \leq_{\text{maj}} \mathbf{Y}$ means that we can go from $\mathbf{X}$ to $\mathbf{Y}$ by hypersequent Curry lemma analogue
4. majoring ordering is a wqo so there exists N such that $S_{N+1} = S_N$
5. Using length function theorem (Balasubramanian 2020) to get max value for N: each FLew extensions under consideration is in $\mathbf{F}_{\omega^\omega}$

↪ Further questions

Further questions

1. Can we find a logic in $\mathbf{F}_{\omega^\omega} - \mathbf{F}_\omega$?

   cut-freeness seems to need hypersequents naturally lead to $\mathbf{F}_{\omega^\omega}$

2. lower bound and sharper upper bounds for MTL

   This was first syntactic proof and first complexity bound for MTL (many would suspect that more modest bounds should hold)

3. Simpler lower bound problem? lower bounds for FLec / FLew +

$$\frac{X, X, Z \Rightarrow F \qquad Y, Y, Z \Rightarrow F}{X, Y, Z \Rightarrow F}\ \text{scom}$$

   'double antecedent, share between premises'

   For example

$$\frac{p, q^4 \Rightarrow \qquad p^3, q^2 \Rightarrow}{p^2, q^3 \Rightarrow}$$

   What type of (counter?) machine could we embed here?

4. Is uninorm logic HFLe + com decidable ?

   Some extensions of HFLe are undecidable: Galatos and St. John, 2021.

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