Fixed-point elimination in Heyting algebras

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[1] Joint work with S. Ghilardi, UniMi, and M. J. Gouveia, ULisboa.
See [Ghilardi et al., 2016, Ghilardi et al., 2019].

Plan

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The intuitionistic μ-calculus

The elimination procedure

Bounding closure ordinals

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μ-calculi

Syntax:

Add to a given algebraic framework
syntactic least and greatest fixed-point constructors.

E.g., the propositional modal μ-calculus:

$$\phi := x \mid \neg x \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \mid \Box\phi \mid \Diamond\phi$$

$$\mid \mu_x.\phi \mid \nu_x.\phi\,, \qquad \text{when } x \text{ is positive in } \phi.$$

Semantics:

Interpret the syntactic least (resp. greatest) fixed-point as expected.

$[\![\mu_x.\phi]\!]_v :=$

least fixed-point of the monotone mapping $X \mapsto [\![\phi]\!]_{v,X/x}$

Alternation hierarchies in μ-calculi

Let $\sharp\phi$ count the number of alternating blocks of fixed-points in ϕ.

Problem. For a given μ-calculus, does there exist n such that, for each ϕ with $\sharp\phi > n$, there exists ψ with $\gamma \equiv \psi$ and $\sharp\psi \leq n$?

Alternation hierarchies, facts

- The alternation hierarchy for the modal μ-calculus is infinite (there exists no such n) [Lenzi, 1996, Bradfield, 1998, Arnold, 1999].
- Idem for the lattice μ-calculus [Santocanale, 2002].
- The alternation hierarchy for the linear μ-calculus ($\Diamond x = \Box x$) is reduced to the Büchi fragment (here $n = 2$) .
- The alternation hierarchy for the modal μ-calculus on transitive frames collapses to the alternation free fragment (here $n = 1.5$) [Alberucci and Facchini, 2009].
- The alternation hierarchy for the distributive μ-calculus is trivial (here $n = 0$) [Kozen, 1983].

Research plan:

Develop a theory explaining why alternation hierarchies collapse.

The distributive μ-calculus

Kozen :

> *We can assume that every bound variable in a formula of the modal μ-calculus is guarded (appears in the scope of a modal operator).*

That's because

$$\mu_x.\phi \equiv \phi(\bot)\,, \qquad \nu_x.\phi \equiv \phi(\top)\,,$$

if ϕ is generated by the grammar

$$\begin{aligned} \phi := x \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \\ \mid \mu_x.\phi \mid \nu_x.\phi \end{aligned} \qquad \text{when } x \text{ is positive in } \phi.$$

That is:
Proposition. The alternation hierarchy of the μ-calculus on distributive lattices is trivial.
I.e.: every lattice μ-term is equivalent, on distributive lattices, to a fixed-point free term.

The μ-calculus on generalized distributive lattices

Theorem. [Frittella and Santocanale, 2014] There are lattice varieties $\mathcal{D}_n$, $n \geq 0$, such that

1. $\mathcal{D}_0$ is the variety of distributive lattices,
2. $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$, for each $n \geq 0$,
3. for every lattice term ϕ,

$$\mathcal{D}_n \models \phi^{n+1}(\bot) = \phi^{n+2}(\bot)\,, \qquad \mathcal{D}_n \not\models \phi^{n}(\bot) = \phi^{n+1}(\bot)\,,$$

   (and duals).

Corollary. The alternation hierarchy of the lattice μ-calculus is trivial on $\mathcal{D}_n$, for each $n \geq 0$.

Plan

The intuitionistic μ-calculus

After the distributive μ-calculus, the next on the list—by Pitt-Visser's quantifiers, we knew that least fixed-points and greatest fixed-points are definable.

We extend the signature of Heyting algebras (i.e. Intuitionistic Logic) with least and greatest fixed-point constructors.

Intuitionistic μ-formulas are generated by the grammar:

$$\begin{aligned}\phi := x \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \mid \phi \to \phi \\ \mid \mu_x.\phi \mid \nu_x.\phi\,, \qquad \text{when } x \text{ is positive in } \phi.\end{aligned}$$

Heyting algebra semantics

We take any *provability* semantics of IL with fixed points:

- (Complete) Heyting algebras.
- Kripke frames.
- Any sequent calculus for Intuitionisitc Logic (e.g. LJ) plus Park/Kozen's rules for least and greatest fixed-points:

$$\frac{\phi[\psi/x] \vdash \psi}{\mu_x.\phi \dashv \psi} \qquad \frac{\Gamma \vdash \phi(\mu_x.\phi)}{\Gamma \vdash \mu_x.\phi} \quad \frac{\phi(\nu_x.\phi) \vdash \delta}{\nu_x.\phi \vdash \delta} \qquad \frac{\psi \vdash \phi[\psi/x]}{\psi \vdash \nu_x.\phi}$$

Definition. A *Heyting algebra* is a bounded lattice $H = \langle H, \top, \wedge, \bot, \vee \rangle$ with an additional binary operation $\to$ satisfying

$$x \wedge y \leq z \quad \text{iff} \quad x \leq y \to z\,.$$

Pitt-Visser's (bisimulation) quantifiers

Proposition. [Pitts, 1992, Ghilardi and Zawadowski, 1997] Given ϕ with $x \in Var(\phi)$, we can construct a term $\exists_x.\phi$ such that

1. $Var(\exists_x.\phi) \subseteq Var(\phi) \setminus \{ x \}$, and
2. whenever $Var(\psi) \subseteq Var(\phi) \setminus \{ x \}$, we have

$$\phi \leq \psi \quad \text{iff} \quad \exists_x.\phi \leq \psi \,.$$

NB: we can also construct $\forall_x.\phi$.

We can define least and greatest fixed points from the quantifiers, e.g.:

$$\mu_x.\phi(x) \equiv \exists_x.(x \wedge x \to \phi(x))\,.$$

Ruitenburg's theorem [Ruitenburg, 1984][2]

Theorem. For each intuitionistic formula ϕ, there exists $n \geq 0$ such that $\phi^n(x) \equiv \phi^{n+2}(x)$.

Corollary. If ϕ is monotone, then, for such n

$$\bot \leq \phi(\bot) \leq \ldots \leq \phi^n(\bot) \leq \phi^{n+1}(\bot) \leq \phi^{n+2}(\bot) = \phi^n(\bot)\,,$$

so $\phi^n(\bot)$ is the least fixed-point of ϕ.

Corollary. The alternation hierarchy for the intuitionistic μ-calculus is trivial.

NB : Ruitenburg's n might not be the *closure ordinal* of $\mu_x.\phi$.

[2] See also [Ghilardi and Santocanale, 2020].

Peirce, compatibility, strenghs, and strongness

Proposition. (Peirce's theorem for Heyting algebras.)
Every formula ϕ is compatible. In particular, for ψ, χ arbitrary formulas, the equation

$$\phi[\psi/x] \wedge \chi = \phi[\psi \wedge \chi/x] \wedge \chi$$

holds on Heyting algebras.

Corollary. Every formula ϕ monotone in x is *strong* in x. That is, any the following equivalent conditions

$$\phi[\psi/x] \wedge \chi \leq \phi[\psi \wedge \chi/x]\,, \qquad \psi \to \chi \leq \phi[\psi/x] \to \phi[\chi/x],$$

hold, for any formulas ψ and χ.

Plan

Greatest fixed-points

Proposition. On Heyting algebras, we have

$$\nu_x.\phi = \phi(\top)\,.$$

Using the deduction theorem and Pitts' quantifiers:

$$\nu_x.\phi(x) = \exists_x.(x \wedge x \to \phi(x)) = \exists_x.(x \wedge \phi(x)) = \phi(\top)\,.$$

Using strongness:

$$\phi(\top) = \phi(\top) \wedge \phi(\top) \leq \phi(\top \wedge \phi(\top)) = \phi^2(\top)\,.$$

Greatest solutions of systems of equations

Proposition. On Heyting algebras, a system of equations

$$\left\{ \begin{array}{ll} x_1 & = \phi_1(x_1, \ldots, x_n) \\ \vdots & \\ x_n & = \phi_n(x_1, \ldots, x_n) \end{array} \right\}$$

has a greatest solution obtained by iterating

$$\phi := \langle \phi_1, \ldots, \phi_n \rangle$$

n times from $\top$.

Proof. Using the *Bekic property*:

$$\mu_x.\langle f(x,y), g(x,y) \rangle = \langle \mu_x.f(x, \mu_y.g(x,y)), \mu_y.g(..,y) \rangle$$

Least fixed-points: splitting the roles of variables

Due to the *diageq*

$$\mu_x.f(x,x) = \mu_x.\mu_y.f(x,y)$$

we can separate computing the least fixed-points w.r.t:

weakly negative variables: variables having an occurrence within the left-hand-side of an implication,

strongly positive variables: those having an occurrence only within the right-hand-side of an implication.

Weakly negative least fixed-points: an example

Use the *roll* equation

$$\mu_x.(f \circ g)(x) = f(\,\mu_y.(g \circ f)(y)\,)$$

to argue that:

$$\begin{aligned}\mu_x.[\,(x \to a) \to b\,] &= [\,\nu_y.(y \to b) \to a\,] \to b \\ &= [\,(\top \to b) \to a\,] \to b \\ &= [\,b \to a\,] \to b\,.\end{aligned}$$

Weakly negative least fixed-points: reducing to greatest fixed-points

If each occurrence of x in ϕ is weakly negative, then

$$\phi(x) = \phi_0[\phi_1(x)/y_1, \ldots, \phi_n(x)/y_n]$$

with $\phi_0(y_1, \ldots, y_n)$ negative in each y_j.

Due to

$$\mu_x.(f \circ g)(x) = f(\, \mu_y.(g \circ f)(y)\,)$$

we have

$$\begin{aligned}\mu_x.\phi(x) &= \mu_x.(\, \phi_0 \circ \langle \phi_1, \ldots, \phi_n \rangle\,)(x) \\ &= \phi_0(\, \nu_{y_1 \ldots y_n}.(\, \langle \phi_1, \ldots \phi_n \rangle \circ \phi_0\,)(y_1, \ldots y_n))\,.\end{aligned}$$

Interlude: least fixed-points of strong functions

If f and f_i, $i \in I$, are strong, then

$$\mu_x.a \wedge f(x) = a \wedge \mu_x.f(x)\,,$$
$$\mu_x.\bigwedge_{i \in I} f_i(x) = \bigwedge \mu_x.f_i(x)\,,$$
$$\mu_x.a \to f(x) = a \to \mu_x.f(x)\,.$$

Strongly positive fixed-points: disjunctive formulas

The equation

$$\mu_x. \bigwedge_{i \in I} f_i(x) = \bigwedge_{i \in I} \mu_x . f_i(x)$$

allows to push least fixed-points down through conjunctions.

Once all conjunctions have been pushed up in formulas, we are left to compute least fixed-points of *disjunctive formulas*, generated by the grammar:

$$\phi = x \mid \beta \vee \phi \mid \alpha \to \phi \mid \bigvee_{i=1,\ldots,n} \phi_i \,,$$

where α and β do not contain the variable x.

We call α an *head subformula* and β a *side subformula*.

Least fixed-points of inflating functions

All functions f denoted by such formula ϕ are (monotone and) *inflating*:

$$x \leq f(x)\,.$$

Let f_i, $i = 1, \ldots, n$, be a collection of monotone inflating functions. Then

$$\mu_x. \bigvee_{i=1,\ldots,n} f_i(x) = \mu_x.(f_1 \circ \ldots \circ f_n)(x)\,.$$

Least fixed-points of disjunctive formulas

Proposition. Let ϕ be a disjunctive formula, with $\mathit{Head}(\phi)$ (resp., $\mathit{Side}(\phi)$) the collection of its head (resp., side) subformulas. Then

$$\mu_x.\phi = \bigwedge_{\alpha \in \mathit{Head}(\phi)} \alpha \to \bigvee_{\beta \in \mathit{Side}(\phi)} \beta\,.$$

If $\mathit{Head}(\phi) = \{\, \alpha_1, \ldots, \alpha_n \,\}$ and $\mathit{Side}(\phi) = \{\, \beta_1, \ldots, \beta_m \,\}$:

$$\begin{aligned}
\mu_x.\phi &= \mu_x.\alpha_1 \to \alpha_2 \to \ldots \to \alpha_n \to \beta_1 \vee \ldots \vee \beta_m \vee x \\
&= \mu_x. \bigwedge_{\alpha \in \mathit{Head}(\phi)} \alpha \to \bigvee_{\beta \in \mathit{Side}(\phi)} \beta \vee x \\
&= \bigwedge_{\alpha \in \mathit{Head}(\phi)} \alpha \to \mu_x. \bigvee_{\beta \in \mathit{Side}(\phi)} \beta \vee x \\
&= \bigwedge_{\alpha \in \mathit{Head}(\phi)} \alpha \to \bigvee_{\beta \in \mathit{Side}(\phi)} \beta\,.
\end{aligned}$$

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Closure ordinals

Definition. (*Closure ordinal*). For $\mathcal{K}$ a class of models and $\phi(x)$ a monotone formula/formula, let

$$\texttt{cl}_{\mathcal{K}}(\phi) = \text{least ordinal } \alpha \text{ such that } \mathcal{K} \models \mu_x.\phi = \phi^{\alpha}(\bot)\,.$$

In general, $\texttt{cl}_{\mathcal{K}}(\phi)$ might not exist.

If $\mathcal{H}$ is the class of Heyting algebras and $\phi(x)$ is an intuitionistic formula, then

$$\texttt{cl}_{\mathcal{H}}(\phi) < \omega\,.$$

Upper bounds from fixed-point equations

$$\boxed{\mu_x.(f \circ g)(x) = f(\,\mu_y.(g \circ f)(y)\,)}$$

$$\mathtt{cl}_{\mathcal{K}}(\,f \circ g\,) \leq \mathtt{cl}_{\mathcal{K}}(\,g \circ f\,) + 1,$$

$$\boxed{\mu_x.f(x,x) = \mu_x.\mu_y.f(x,y)}$$

$$\mathtt{cl}_{\mathcal{K}}(\,f \circ \Delta\,) \leq n \cdot \mathtt{cl}_{\mathcal{K}}(g),$$

with $n = \mathtt{cl}_{\mathcal{K}}(f(x,\_))$ and $g(x) = \mu_y.f(x,y)$,

$$\boxed{\mu_x.\langle f(x,y), g(x,y)\rangle = \langle \mu_x.f(x, \mu_y.g(x,y)), \mu_y.g(..,y)\rangle}$$

$$\mathtt{cl}_{\mathcal{K}}(\,\langle f,g\rangle\,) \leq (\mathtt{cl}_{\mathcal{K},y}(g)+1)(\mathtt{cl}_{\mathcal{K}}(h)+1) - 1,$$

with $h(x) = f(x, \mu_y.g(x,y))$.

Other bounds

$$\texttt{cl}_{\mathcal{H}}(\,\phi_0(\phi_1(x),\ldots,\phi_n(x))\,) \leq n+1, \qquad \text{when } \phi_0(y_1,\ldots,y_n) \text{ contravariant,}$$

$$\texttt{cl}_{\mathcal{H}}(\,\phi\,) \leq \mathbf{card}(\mathit{Head}(\phi))+1\,,$$
when ϕ is a disjunctive formula,

$$\texttt{cl}_{\mathcal{K}}(f \wedge g) \leq \texttt{cl}_{\mathcal{K}}(f) + \texttt{cl}_{\mathcal{H}}(g) - 1,$$
when f and g are strong.

Back to Ruitenburg's theorem

- Let $\rho(\phi)$ be the least n such that $\phi^{n+2} \equiv \phi$.
- Inspection of Ruitenburg's paper shows that

$$\mathtt{cl}_{\mathcal{H}}(\phi) \leq \rho(\phi) = O(n)\,,$$

where n is the number of implication symbols in ϕ.

- Given a strongly positive formula ϕ, pushing up conjuctions yields a formula

$$\bigwedge_{i=1,\ldots,k} \delta_i\,, \quad \delta_i \text{ disjunctive,}$$

where k might be exponential w.r.t. the size of ϕ.

An naive application of the procedure yields the upper bound

$$\mathtt{cl}_{\mathcal{H}}(\phi) = \mathtt{cl}_{\mathcal{H}}(\bigwedge_{i=1,\ldots,k} \delta_i) \leq 1 + \sum_{i=1,\ldots,k} (\mathtt{cl}_{\mathcal{H}}(\delta_i) - 1) \leq 1 + k(N-1)\,,$$

of N is an upper bound for $\mathtt{cl}_{\mathcal{H}}(\delta_i)$, for each $i = 1, \ldots, k$.
This might be exponential w.r.t. the size of ϕ.

Closing the gap: Ruitenburg's number for strongly positive formulas

Theorem. Let $\delta_1, \delta_2, \ldots, \delta_n$ be disjunctive formulas and put

$$\phi(x) := \bigwedge_{i=1,\ldots,n} \delta_i(x)\,.$$

Then

$$\mathtt{cl}_{\mathcal{H}}(\phi) \leq \rho(\phi) \leq (N+1)(M+1)$$

where N is the number of distinct head subformulas in any of the δ_i and M is the number of distinct side subformulas occurring in any of the δ_i.

Proof: by studying algebraic properties of disjunctive formulas whose head formulas are from $\mathcal{A}$ and side formulas are from $\mathcal{B}$.

. . . and then, gaming with conjunctions of such formulas of iterated such conjunctions.

Remark. Upper bound orthogonal to Ruitenburg's bound.

Studying disjunctive formulas

Disjunctive formulas over $\mathcal{A}$ and $\mathcal{B}$:

$$\phi \Rightarrow x \mid [A]\phi \mid \bigvee B \vee \phi \mid \phi \vee \phi \,,$$

where $A \subseteq \mathcal{A}$ and $B \subseteq \mathcal{B}$ are sets of propositional variables, and

$$[A]\phi := (\bigwedge A) \to \bigvee \phi \,.$$

Letter formula and word formulas:

$$\phi_{(A,B)} := [A](\bigvee B \vee x) \qquad \phi_{(A,B)\cdot w} := \phi_{(A,B)} \circ \phi_w \qquad (= \phi_{(A,B)}[\phi_w/x]\,)\,.$$

Properties:

$$\begin{aligned}
\phi_{(A_0,\emptyset)} \circ \phi_{(A_1,B_1)} &= \phi_{(A_0 \cup A_1, B_1)} \\
\phi_{(A_0,B_0)} \circ \phi_{(\emptyset,B_1)} &= \phi_{(A_0, B_0 \cup B_1)} \\
\phi_{(A_0,B_0)} \circ \phi \circ \phi_{(A_1,B_1)} &= \phi_{(A_0, B_0 \setminus B_1)} \circ \phi \circ \phi_{(A_1 \setminus A_0, B_1)} \\
\rho(\phi_w) \leq 2 \qquad & \phi^{\rho(\phi)} = \phi_{Supp(\phi)} \,,
\end{aligned}$$

where $Supp(\phi)$ is the pair (A, B) of propositonal variables appearing in ϕ.

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1. Weakly negative x:

$$\bigwedge_{i=1,\dots,n} (x \to a_i) \to b_i$$

converges after $n+1$ steps. This upper bound is strict.

2. Strongly positive x:

$$b \vee \bigvee_{i=1,\dots,n} a_i \to x$$

converges after $n+1$ steps. This upper bound is strict.

Examples (II)

- Similarly,

$$\phi(x) := \bigvee_{i=1,\dots,n} (x \to a_i) \to b_i$$

converges within $n+1$ steps, according to the general theory.

Theorem. For any $n \geq 2$, $\phi(x)$ converges to its least fixed-point within 3 steps.

Thanks to

fCube, `http://www2.disco.unimib.it/fiorino/fcube.html`, [Ferrari et al., 2010]

Questions

- Do the general bounds on finite closure ordinals apply also in the infinite case? Are these bounds of any use (e.g. for closure ordinals in the modal μ-calculus)?

- Is there a formula ϕ such that $\mathtt{cl}_{\mathcal{H}}(\phi) < \rho(\phi)$?

- Can we generalize the elimination procedure to derive Ruitenburg's theorem?

- Given a class of formulas (described in some way), can we decide a uniform upper bound on its set of closure ordinals?

Thanks ! Questions ?

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