Theory and Applications of Craig Interpolants

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Craig Interpolation

A Craig interpolant of φ, ψ is a formula χ in the shared signature of φ and ψ with $\varphi \models \chi \models \psi$.

The Craig interpolation property (that Craig interpolants exist whenever $\varphi \models \psi$) was shown for FO by Craig in the 1950s.

State of the Art in 2008 in Special Issue of Synthese:

- Feferman, Väänänen: mathematical logic, in particular abstract model theory
- Demopoulos, M Friedman: Philosophy of Science
- Tinelli, de Lavalette: Verification and modular software specification
- D'Agostino: modal and non-classical logic
- van Benthem: fragments of FO and other aspects

Craig Interpolation

Workshop series iPRA , `https://ipra-2022.bitbucket.io` mostly work in computer science:

- verification (interpolation in SAT, QBF, and many weak theories)
- automated deduction (interpolants from resolution and other proofs in FO)
- databases (interpolants for query reformulation, generating plans for query execution)
- knowledge representation (modular knowledge bases, query reformulation)

My plan

- Interpolants in propositional logic: uniform interpolants, Beth definability, size of interpolants.
- Craig interpolants in FO: uniform interpolants, separation, failure on finite models.
- Craig interpolation property (CIP) in modal logic: proofs of CIP using bisimulations, computing uniform interpolants.
- What to do without CIP? (Mainly for modal logics.)

Basic Definitions

Given formulas φ, ψ, a formula χ is called a Craig interpolant of φ, ψ if

- $\varphi \models \chi$ and $\chi \models \psi$;
- $\text{sig}(\chi) \subseteq \text{sig}(\varphi) \cap \text{sig}(\psi)$.

In propositional logic, $\text{sig}() = \text{atom}()$.

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- $p \wedge q_1 \models q_2 \rightarrow p$. Craig interpolant: p.
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Examples.

- $p \wedge q_1 \models q_2 \rightarrow p$. Craig interpolant: p.
- $p \wedge \neg p \models q$. Craig interpolant: $\bot$. (Having constants for true/false is important. Without CIP does not hold for formulas in disjoint signatures).

Proof

QBF (quantified boolean formulas) are an extension of propositional logic with quantifiers over propositional atoms:

$$\varphi, \psi = p \mid \text{true} \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists p.\varphi$$

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Satisfaction of φ under a valuation v into $\{0, 1\}$, $v \models \varphi$, is defined inductively as usual with

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The signature $\text{sig}(\varphi)$ is defined inductively as expected with $\text{sig}(\exists p.\varphi) = \text{sig}(\varphi) \setminus \{p\}$.

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So $\exists\mathbf{p}.\varphi$ is a Craig interpolant of φ, ψ, but in QBF and not in propositional logic.

As propositional logic is functionally complete there exists a propositional formula χ with $\text{sig}(\chi) = \text{sig}(\exists\mathbf{p}.\varphi)$ such that $\chi \equiv \exists\mathbf{p}.\varphi$. χ is as required.
Note: we have also proved that QBF trivially has CIP.

A few observations

- Instead of $\exists \mathbf{p}.\varphi$ we could have also used $\forall \mathbf{q}.\psi$ for $\mathbf{q} = \text{sig}(\psi) \setminus \text{sig}(\varphi)$.

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- (The formula equivalent to) $\exists \mathbf{p}.\varphi$ is the logically strongest interpolant (it entails all others) and $\forall \mathbf{q}.\psi$ is the logically weakest interpolant (it is entailed by all others).
- (The formula equivalent to) $\exists \mathbf{p}.\varphi$ does not depend on ψ, but only on **p**. So it works for any ψ' with $\varphi \models \psi'$ and $\mathbf{p} \cap \text{sig}(\psi') = \emptyset$. These are also known as uniform interpolants.
- Note that QBF trivially always has uniform interpolants.

Implicit/Explicit Definability

Let Σ be a set of atoms and $p \notin \Sigma$.

p is implicitly Σ-definable under φ if for any valuations v_1, v_2 satisfying φ:

$$v_1(q) = v_2(q) \text{ for all } q \in \Sigma \text{ implies } v_1(p) = v_2(p)$$

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Clearly explicit definability implies explicit definability. The converse is called projective Beth definability property (BDP).

“Craig” implies “Beth”

Assume p is implicitly Σ-definable under φ. Let φ_1 and φ_2 be obtained from φ by replacing symbols q not in Σ by copies q_1 and q_2, respectively. Then implicit definability implies

$$\varphi_1 \wedge \varphi_2 \models p_1 \leftrightarrow p_2$$

Hence

$$\varphi_1 \wedge p_1 \models \varphi_2 \rightarrow p_2$$

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Any interpolant χ of $\varphi_1 \wedge p_1, \varphi_2 \rightarrow p_2$ is a Σ-definition of p under φ.

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(Note: Define the size of a formula as the number of its subformulas. So use the representation of a formula as a DAG, not a tree.)

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$\wedge, \vee$-Interpolants

Rather deep results on the size of interpolants are known, however, if we consider interpolants in the language with

$$\wedge, \quad \vee, \quad \top, \quad \bot$$

simply called $\wedge, \vee$-interpolants.

Makes sense only if we know already that the interpolants are monotone (if a truth value moves from 0 to 1, the truth value of the formula cannot move from 1 to 0).

$\wedge, \vee, \top, \bot$ are functionally complete for monotone functions.

No poly-size $\wedge, \vee$-uniform interpolants

Idea: define formula $\exists \mathbf{q}.C_n^k$ that says that a size n graph encoded by atoms $\mathbf{p} = p_{ij}$, $i, j \in [n]$ has a clique of size k.

$\exists \mathbf{q}.C_n^k$ is monotone, but

Theorem (Razborov 1985). No $\wedge, \vee$-formula equivalent to $\exists \mathbf{q}.C_n^k$ is of polynomial size.

A few more details...

No poly-size $\wedge, \vee$-uniform interpolants

Encode undirected graphs with n nodes $[n] = \{0, 1, \ldots, n-1\}$ using atoms $\mathbf{p} = p_{ij}$, $i, j \in [n]$, indicating an edge between i and j.
Using 'helper symbols' $\mathbf{q} = q_{iv}$, $i \in [n], v \in [k]$, define C_n^k such that $\exists\mathbf{q}.C_n^k$ says 'graph contains a k-clique':

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q_{iv} says that i is the vth clique member, so we add

$$\bigvee_{i \in [n]} q_{iv}, \text{for } v \in [k]$$

(some i must be the vth clique member) and

$$\neg q_{iv} \vee \neg q_{i'v}, \text{ for } i \neq i'$$

(not two i, i' can be the vth clique member) and

$$(q_{iv} \wedge q_{i'v'}) \to p_{i,i'}$$

(i, i' are not both in clique if $(i, i') \notin E$.)

No poly-size $\wedge, \vee$ Craig interpolants

Let $\exists \mathbf{q}.C_n^k$ say that graph encoded by $\mathbf{p}$ contains a k-clique.

Let $\exists \mathbf{r}.D_n^k$ say graph is k-colorable using 'helper symbols' $\mathbf{r} = r_{ij}$, $i \in [k], j \in [n]$ (r_{ij} says that j has color i). Then

$$C_n^k \models \neg D_n^{k-1}$$

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Theorem (Alon and Boppana 1987). No $\wedge, \vee$-interpolant is of poly-size.

Interpolants as tool for lower bounds of proof length

Proof system that admits construction of interpolants from proofs in poly- time has feasible interpolation (Krajicek 1997).

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Remark 1. For Frege systems feasible interpolation is open (depends of cryptographic assumptions).

Remark 2. Relevance of feasible interpolation for model checking first observed by McMillan 2005.

First-order Logic: Craig's Theorem

In the 1950s, Craig proved that FO has CIP: for any FO-formulas φ, ψ with $\varphi \models \psi$ there exists a formula χ with

$$\text{sig}(\chi) \subseteq \text{sig}(\varphi) \cap \text{sig}(\psi)$$

such that $\varphi \models \chi$ and $\chi \models \psi$. Here sig(χ) is the set of relation and function symbols in χ.

According to (Craig 2008), Craig first did not find this result very interesting without additional constraints on the shape of χ.

According to (van Benthem 2008), Craig was even not interested in Craig interpolation first, but in uniform interpolation.

Craig's Motivation from Philosophy (I guess)

Two assumptions (possibly unrealistic):

- A significant part of physics can be formulated as a finitely axiomatized first-order theory T.
- The signature S of T can be partitioned into two sets S_{theory} and S_{obs} of theoretical and observational terms.

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In other words, does there exist a finite set T_{obs} such that

- $\text{sig}(T_{obs}) \subseteq S_{obs}$;
- $T \models T_{obs}$;
- If $T \models \varphi$ and $\text{sig}(\varphi) \cap S_{theory} = \emptyset$, then $T_{obs} \models \varphi$.

Answer: No

Let T be axiomatized as

$$\forall x\ A(x) \rightarrow B(x), \quad \forall x\ B(x) \rightarrow \exists y\ (r(x,y) \wedge B(y))$$

and $S_{theory} = \{B\}$, $S_{obs} = \{r, A\}$. There does not exist a T_{obs} with the required properties because it would have to imply for all n:

$$T_{obs} \models A(x_0) \rightarrow \exists x_1 \cdots \exists x_n\ \ r(x_0, x_1) \wedge \cdots r(x_{n-1}, x_n)$$

Uniform Interpolants

A formula χ is called a uniform interpolant for φ and $\Sigma \subseteq \text{sig}(\varphi)$ if it is an interpolant for φ, ψ whenever

- $\varphi \models \psi$;
- $\text{sig}(\varphi) \cap \text{sig}(\psi) \subseteq \Sigma$;
- in particular, $\text{sig}(\chi) \subseteq \Sigma$.

A logic for which uniform interpolants exist for all φ, Σ has uniform interpolation.

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Proof. Given φ and Σ, take $\exists \mathbf{X}.\varphi$ were $\mathbf{X} = \mathrm{sig}(\varphi) \setminus \Sigma$.

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Lots of research on uniform interpolants in knowledge representation and reasoning (KR) for decidable fragments of FO.

Intermezzo: KR and uniform interpolants

In KR, uniform interpolation of interest because theories T can be very large (more than 300 000 axioms) but applications often require its content for a small signature Σ only.

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Typical KR languages do not enjoy uniform interpolation, but in practice they still mostly exist. So work on deciding whether uniform interpolants exists and computing it if it does.

$$T = \{\Box_u(A \rightarrow B), \Box_u(B \rightarrow \Diamond_R B)\}$$

Craig Interpolation as Separation

- A class K of models is called elementary if it is the class of models of an FO-sentence φ.
- K is called pseudo-elementary if it is the class of models of a second-order sentence $\exists\vec{S}.\varphi$, where φ is a FO-sentence. (In order words: K is the class of reducts without $\vec{S}$-interpretations of models of φ.)

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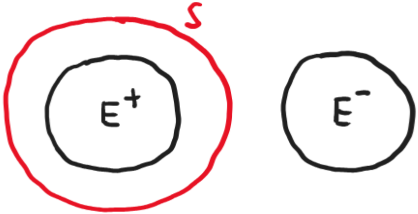
Example: the class of models $M = (D, A^M, r^m)$ in which for each A-node d there is a sequence $d = d_0 r^M d_1 r^M d_2 \cdots$ is pseudo-elementary and not elementary.

Craig Interpolation as Separation

Craig Interpolation is equivalent to: for any disjoint pseudo-elementary classes E^+ and E^- there exists a separating elementary class S, i.e.,

$$E^+ \subseteq S, \quad E^- \cap S = \emptyset$$



Craig Interpolation as Separation

To prove the equivalence, assume

$$E^+ = Mod(\exists X_1.\varphi_1), \quad E^- = Mod(\exists X_2.\varphi_2)$$

and $E^+ \cap E^- = \emptyset$. Then

$$\models \exists X_1.\varphi_1 \to \neg\exists X_2.\varphi_2$$

which is equivalent to (assuming X_1, X_2 disjoint sets of relation symbols)

$$\models \varphi_1 \to \neg\varphi_2$$

Take a Craig interpolant ψ for $\varphi_1, \neg\varphi_2$. Then

$$E^+ \subseteq Mod(\psi), \quad Mod(\psi) \cap E^- = \emptyset$$

Craig interpolation as FO $= \Sigma^1_1 \cap \Pi^1_1$

In words: if

- $\varphi \equiv \exists X_1 . \psi_1$ and
- $\varphi \equiv \forall X_2 . \psi_2$
- with ψ_1, ψ_2 FO

then φ is FO.

Proof. Direct consequence of above for E^- complement of E^+.

FO on finite models does not have CIP

Let $\varphi_{<,A}$ state

- $<$ is a strict linear order on the domain, $A(x)$ holds at its first node and then at exactly every second node, but not in its final node. If $M \models \varphi_{<,A}$, then $|M|$ is even.

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- $<'$ is a strict linear order on the domain, $A'(x)$ holds at its first node and then at every second node, and in its final node. If $M \models \varphi_{<',A'}$, then $|M|$ is odd.

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Hence $\varphi_{<,A} \models \neg\varphi_{<',A'}$. There exists no Craig interpolant since that would have to be true in exactly all models with an even number of points.