Variable-hypothetical conditionals

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This work develops a probabilistic semantics and logic for conditionals based on an unexplored intuition by F.P. Ramsey, according to which conditionals should be understood as instances of suitable generalizations, which he calls *variable hypotheticals* ([4], [5], [7]). According to Ramsey, a variable hypothetical is a rule that guides the formation of one's judgements and beliefs: «If I meet a φ I shall judge it as a ψ» (see [3], [6]). Amongst other things, variable hypotheticals determine the meaning of conditionals and explain their acceptability. In this work, we systematically develop Ramsey's intuition into a formal semantics. The result is a novel and appealing logic of conditionals, that can be applied to both indicatives and counterfactuals.

1 Introduction and heuristics

Consider a conditional such as 'if Tweetie is a bird, then Tweetie flies', formalized as $B(t) \rightarrow F(t)$. Let A be an arbitrary agent. Under ordinary circumstances, $B(t) \rightarrow F(t)$ would seem to be highly acceptable, so let's suppose that A assigns it probability k, i.e. $\mathsf{cr}(B(t) \rightarrow F(t)) = k$, for $k \in [0,1]$ and relatively high. Assigning a value to a conditional, presumably, does not happen in the void. More specifically, when one considers a conditional sentence, several pieces of background information seem relevant to determine its acceptability. To continue with our example, A's assignment of probability k to $B(t) \rightarrow F(t)$ arguably presupposes several pieces of background information – viz. that Tweetie is not a penguin, that Tweetie is not an ostrich, that it is not a chick, ... Let's abbreviate the conjunction of these sentences about Tweetie as $\mathsf{R}(t)$. We can make all this implicit information explicit, and add it to antecedent of the conditional, which then assumes the following form: $B(t) \wedge \mathsf{R}(t) \rightarrow F(t)$. Let's call the latter conditional the *extended version* of the former, or *extended conditional*. By contrast, let's refer to the original conditional as the *simple conditional*.

It seems plausible to suppose that it is the *context* that determines what background information is relevant. Here, we understand 'context' rather broadly, as to specify time, place, agent(s), possible world, and possibly other parameters, together with the pieces of knowledge and belief that agents have. In each context so understood, a simple conditional is associated with exactly one extended version, that is determined by the relevant background information.

Since, we are assuming, it's the extended conditional that agents have in mind, if implicitly, when assessing a simple conditional, we assume that the degree of acceptability of a simple conditional is always identical to that of its extended version, i.e. $\mathsf{cr}(B(t) \rightarrow F(t)) = \mathsf{cr}(B(t) \wedge \mathsf{R}(t) \rightarrow F(t))$.

But what determines the probability of an extended conditional? According to the *variable hypothetical intuition*, whatever degree of belief $B(t) \rightarrow F(t)$ and $B(t) \wedge \mathsf{R}(t) \rightarrow F(t)$ have comes from the 'value' that the corresponding variable hypothetical has. Variable hypotheticals are rules of judgement that express the *objective chance* of an event relative to another. Variable hypotheticals are formed via habits that have proven to be reliable, i.e. they have proven to lead to true beliefs most of the time. In other words, reliable habits produce degrees of belief that closely correspond to, or even perfectly match, the objective chance of a conditional event.

There are several ways to formalize objective chances of conditional events, here we will just assume that variable hypotheticals have objective chances associated with them, without committing ourselves to a specific interpretation of them. We propose to model variable hypotheticals as conditional events involving formulae with an open variable, writing them as $\psi(x)|\varphi(x)$ (where both $\varphi(x)$ and $\psi(x)$ are conditional-free). We also model their objective chance via a primitive function Ch, writing it as

$$\mathsf{Ch}(\varphi(x), \psi(x))$$

We assume that Ch is a probability function, and in particular that it is defined on the value space $[0,1]$. Terminologically, we call any extended conditional $B(t) \wedge \mathsf{R}(t) \to F(t)$ an *instance* of the variable hypothetical $(F(x)|B(x) \wedge \mathsf{R}(x))$.

In this account, variable hypotheticals do the heavy lifting, in that they determine the degree of belief of the corresponding extended and simple conditionals. More specifically, in order to assign a degree of belief to a simple conditional, one assumes a primitive objective chance for variable hypotheticals, and considers the variable hypothetical associated with the extension of that conditional. Finally, one supposes that the degree of belief of both the simple and extended conditional associated with that variable hypothetical equals the objective chance of the latter. Of course, we are considering an ideal agent, whose credences perfectly match objective chances; in a more realistic scenario, an agent's credences might deviate from objective changes. Still, in the variable hypothetical accounts, the degree of belief of a simple conditional is always identical to the degree of belief of its extension, and both are determined by the objective chances of the associated variable hypothetical.

To continue with our running example, this means that, for every term t:

$$\begin{aligned} \mathsf{Ch}(F(x)|B(x) \wedge \mathsf{R}(x)) &= \mathsf{cr}(B(t) \wedge \mathsf{R}(t) \to F(t)) \\ &= \mathsf{cr}(B(t) \to F(t)) \end{aligned}$$

This is why, in our reconstruction, an agent A assigns degree of belief k to the conditional $B(t) \to F(t)$ in context c. In a context c, an agent A (implicitly) associates the simple conditional $B(t) \to F(t)$ with its extended version $B(t) \wedge \mathsf{R}(t) \to F(t)$, and hence with the variable hypothetical $(F(x)|B(x) \wedge \mathsf{R}(x))$. The latter is assigned an objective chance, which is then inherited as the degree of belief of both the extended and the simple conditional.

So far, we have outlined Ramsey's intuition about variable hypotheticals and conditionals. We now set out to make it more formally precise.

2 Probabilistic semantics

We work in a standard first-order language $\mathscr{L}$, whose vocabulary includes countably many individual constants (c_0, c_1, ...), individual variables (x, y), predicates (P, Q, R, ...) of each arity. The logical vocabulary of $\mathscr{L}$ includes $\neg$, $\vee$, and $\forall$ (while $\wedge$, $\supset$, and $\exists$ are defined as usual), together with a primitive symbol for the non-material conditional ($\to$). Finally, $\mathscr{L}$ includes also auxiliary symbols (such as brackets and commas). $\mathscr{L}$-formulae are inductively defined as usual. Closed formulae are called sentences.

Even though $\mathscr{L}$ is a standard first-order language, with an extra connective, we will not provide a semantics (and a logic) for all its formulae. Rather, we will target the fragment of $\mathscr{L}$-sentences, called F, that only includes atomic $\mathscr{L}$-sentences and their closure under Boolean operations and at most one occurrence of the conditional. More formally, F is the smallest set of $\mathscr{L}$-sentences such that $\varphi \in$ F just in case:

- φ is an atomic $\mathscr{L}$-sentence, or

- φ is $\neg\psi$ and $\psi \in \mathsf{F}$ and conditional-free, or

- φ is $\psi \vee \chi$ and $\psi \in \mathsf{F}$ and $\chi \in \mathsf{F}$, and they are both conditional-free, or

- φ is $\psi \rightarrow \chi$ and $\psi \in \mathsf{F}$ and $\chi \in \mathsf{F}$, and they are both conditional-free.

For instance, $P(s) \rightarrow (R(t_1,t_2) \wedge Q(s))$ is a formula of F, while neither $P(s) \wedge (R(t_1,t_2) \rightarrow Q(s))$ nor $P(s) \rightarrow (R(t_1,t_2) \rightarrow Q(s))$ are. We now employ conditionals in F in order to formalize variable hypotheticals. Let $\varphi \rightarrow \psi$ be in F, and consider (for $k \leq n$):

$$\psi([x_1/s_{i_1}]\dots[x_n/s_{i_n}])|\varphi([x_1/s_{i_1}]\dots[x_n/s_{i_n}]), \tag{1}$$

where $s_{i_1},\dots,s_{i_n}$ are some of the constants appearing (in a fixed, non-repeating order) in $\varphi \rightarrow \psi$. We use $\psi([x_1/s_{i_1}]\dots[x_n/s_{i_n}])$ to denote the result of uniformly substituting some terms in ψ with fresh open variables. Sentences of the form (1) will be called 'variable hypotheticals', and V will denote the set of all variable hypotheticals.

Conditional sentences are instances of variable hypotheticals. An agent accepts a conditional if its extended version is an instance of a variable hypothetical the agent accepts. The degree to which a conditional is accepted is determined by the probability assignment of the variable hypothetical that it instantiates. Hence, we model the acceptability of a conditional via probability and assume primitive probability assignments attached to variable hypotheticals.

Definition 2.1. cr *is a probability assignment, i.e. a function that respects the Kolmogorov axioms:*

1. $\mathsf{cr} : \mathsf{F} \longmapsto [0,1]$

2. If $\models_{\mathsf{Cl}} \varphi$*, then* $\mathsf{cr}(\varphi) = 1$

3. If φ *and* ψ *are conditional-free and incompatible, then* $\mathsf{cr}(\varphi \vee \psi) = \mathsf{cr}(\varphi) + \mathsf{cr}(\psi)$.

Furthermore, in the present account we have assumed that:

4. For every $B(t) \wedge \mathsf{R}(t) \rightarrow F(t) \in \mathsf{F}$, and every individual constant t,

$$\begin{aligned}\mathsf{Ch}(F(x)|B(x) \wedge \mathsf{R}(x)) &= \mathsf{cr}(B(t) \wedge \mathsf{R}(t) \rightarrow F(t)) \\ &= \mathsf{cr}(B(t) \rightarrow F(t))\end{aligned}$$

We now construct probabilistic models for sentences in F. First, we assume a (countable) model $\mathscr{M}$ for the $\rightarrow$-free part of $\mathscr{L}$. Then, we assume a set C of contexts c_0, c_1, ... s.t. for each context c_j and every simple conditional $B(x) \rightarrow F(x) \in \mathsf{F}$, c_j associates with $B(x) \rightarrow F(x)$ a unique R s.t. $B(t) \wedge \mathsf{R}(t) \rightarrow F(t)$ is an instance of a variable hypothetical in V. Finally, we assume a primitive chance function for the variable hypothetical, as well as probability assignments to all the sentences in F that satisfy our axioms $1-4$.

3 The logic of variable hypothetical conditionals

We then use our probabilistic models to isolate a notion of logical consequence, and obtain a conditional logic. There are several ways to associate our models with notions of logical consequence. Here we explore a standard probabilistic consequence relation, that is Adams's *uncertainty minimization* ([2], [1]), with a slight twist, namely the preservation of the Rs (that is, the background information) of the premises. Let us define both components of our notion of consequence in turn.

First, define the uncertainty of a sentence φ in $\mathsf{V} \cup \mathsf{F}$ as customary, i.e. as $\mathsf{U}(\varphi) = 1 - \mathsf{cr}(\varphi)$, and the uncertainty of a set of sentences Γ in $\mathsf{V} \cup \mathsf{F}$ as $\mathsf{U}(\Gamma) = \mathsf{U}(\gamma_1) + \ldots + \mathsf{U}(\gamma_n)$ for $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. Second, for an inference from Γ to φ, let's define the formula φ_Γ that results from φ by adding to its antecedent (if φ is a conditional, otherwise the definition does not apply) all the Rs of the extended conditionals in Γ. So, for every $\varphi \in \mathsf{F}$, if φ is a conditional, φ_e indicates its extension. (If φ is not a conditional, let φ_e be simply φ.) For every $\Gamma \subseteq \mathsf{F}$, let Γ_e be the set of the extensions of the sentences in Γ, i.e. $\{\gamma_e \,|\, \gamma \in \Gamma\}$. Let φ be a conditional $\psi \to \chi$, and let φ_e be $\psi \wedge \mathsf{R} \to \chi$. For every $\Gamma \subseteq \mathsf{F}$, φ_Γ is defined as $\psi \wedge \mathsf{R} \wedge \mathsf{R}_1, \ldots, \mathsf{R}_n \to \chi$, where $\mathsf{R}_1, \ldots, \mathsf{R}_n$ are the Rs in Γ_e. If φ is not a conditional, φ_Γ is simply φ.

With the notion of uncertainty and φ_Γ at hand, we are finally in a position to define the notion of validity associated with variable hypothetical conditionals. More specifically:

$$\Gamma \vDash_{\mathsf{VH}} \varphi \text{ if and only if } \mathsf{U}(\Gamma_e) \geq \mathsf{U}(\varphi_\Gamma).$$

We conclude by proving that our conditional logic VH has several desirable results. First, it invalidates the paradoxes of material implication:

$$\neg\varphi \nvDash_{\mathsf{VH}} \varphi \to \psi \qquad\qquad \psi \nvDash_{\mathsf{VH}} \varphi \to \psi$$

Then, it validates desirable inferences, such as *modus ponens*:

$$\varphi \to \psi, \varphi \vDash_{\mathsf{VH}} \psi$$

Finally, it invalidates implausible conditional inferences, such as strengthening of the antecedent, transitivity, and contraposition:

$$\varphi \to \chi \nvDash_{\mathsf{VH}} \varphi \wedge \psi \to \chi \qquad\qquad \varphi \to \psi, \psi \to \chi \nvDash_{\mathsf{VH}} \varphi \to \chi \qquad\qquad \varphi \to \psi \nvDash_{\mathsf{VH}} \neg\psi \to \neg\varphi$$

These and other results about the properties of VH show it to be a promising candidate for the logic of indicative (and possibly counterfactual) conditionals, that make it able to compete with several contemporary conditional logics.

References

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