Partially simple graphs form a quasitopos

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1 Introduction

Presheaf categories encompass many mathematical structures and can be used to describe a large variety of categories of graphs, such as multigraphs, reflexive graphs, symmetric graphs, undirected graphs, hypergraphs [9, Example 43]. Presheaf categories are toposes [4], which are categories that can be studied for their internal logic [5].

However, certain categories of graphs, such as simple graphs, are not toposes and thus not presheaves. Simple graphs nonetheless form a *quasitopos* [10], which is a generalisation of the notion of a topos [11]. We have recently shown that also *simple fuzzy graphs* form a quasitopos [9], extending the method used in [10]. In the present abstract, we continue this line of work and prove that *partially simple graphs* form a quasitopos. By partially simple graph, we mean a graph with two sets of edges, one where parallel edges are allowed and one where they are not. The method employed for those proofs uses the notions of a topology on a topos, of density, and of separated elements [4, V.1]. We use the fact that the elements that are separated with respect to a topology form a full subcategory which is a quasitopos [3, Thm. 10.1].

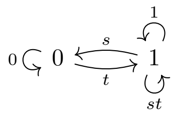
Knowing when a category is a quasitopos is relevant in graph rewriting, for instance. Recently, we have introduced the notion of a *fuzzy presheaf*, which consists of a presheaf $A \in \mathsf{Set}^{I^{\mathrm{op}}}$, where every element $a \in A(i)$ for all $i \in I$ has a membership value inside a poset $(\mathcal{L}(i), \leqslant)$. This includes, for instance, the notion of weighted graphs. Given a small category I, we have shown that when each $(\mathcal{L}(i), \leqslant)$ is a complete Heyting algebra, then the category of fuzzy presheaves is a quasitopos [9]. Having a fuzzy structure lends itself well for implementing relabelling of graphs [7, 8]. Furthermore, quasitoposes have been proposed as a natural setting for non-linear rewriting [1]. They moreover provide a framework to compare and unify algebraic graph rewriting formalisms [8, Theorem 73]. Being in an (rm-adhesive) quasitopos also ensures that certain termination methods can be applied [6]. More theoretically, a logic similar to intuitionistic logic can be studied inside quasitoposes [11, Chapter 3].

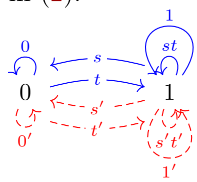
We recall briefly the definitions of a topos and a quasitopos. A category is a ***topos*** if it admits all finite limits, has a subobject classifier, and is cartesian closed. As a consequence of that, a topos also admits all finite colimits and is locally cartesian closed. A ***quasitopos*** is required to have all the aforementioned properties cited, with the sole difference of having only a *regular*-subobject classifier Ω. This means that for all *regular* subobject $m : A \rightarrowtail B$ there is a characteristic morphism $\chi_A : B \to \Omega$ such that the square on the right is a pullback.

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathsf{True}"] \\ B \arrow[r, "\chi_A"'] & \Omega \end{tikzcd}$$

2 Logic and topologies

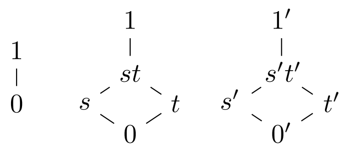
The category Graph consists of (multi)graphs and graph homomorphisms, and is the presheaf category $\mathsf{Set}^{I^{\mathrm{op}}}$ for $I^{\mathrm{op}} = \begin{tikzcd} E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & V \end{tikzcd}$. A graph G consists of an edge set $G(E)$, a vertex set $G(V)$, and a source and a target function $G(s), G(t) : G(E) \to G(V)$. In the same vein, we define BiColGraph, the category of graphs with edges partitioned into two sets. We visualise this partition by using two colours: blue and red. It is the presheaf category for $I^{\mathrm{op}} = \; E \overset{s}{\underset{t}{\rightrightarrows}} V \overset{s'}{\underset{t'}{\leftleftarrows}} E'$. The category Graph has the classifying object Ω described in (1), see e.g. [10]. Analogously, BiColGraph has Ω described in (2).


(1)


(2)

Let us recall the internal logic that lies in a topos [2, 5]. $\mathsf{True} : 1 \to \Omega$ is part of the definition of a subobject classifier. Then, the logical connectives $\mathsf{False} : 1 \to \Omega$, negation $\neg : \Omega \to \Omega$, conjunction, implication and disjunction $\wedge, \Rightarrow, \vee : \Omega \times \Omega \to \Omega$ can all be defined as characteristic functions of some morphisms, see e.g. [2, p.136-139].

Lemma 1. *For the subobject classifier Ω in* Graph*, its edge set $\Omega(E)$ and its vertex set $\Omega(V)$ are Heyting algebras. Similarly, for the subobject classifier Ω in* BiColGraph*, its edge sets $\Omega(E)$ and $\Omega(E')$, and its vertex set $\Omega(V)$ are Heyting algebras, and for $\Omega(E)$ and $\Omega(V)$ they are the same as in* Graph*. Their Hesse diagrams are shown on the right.*



Vigna [10] observed that the logical conjunction coincides with the meet of those Heyting algebras. This is also true for the other logical connectives in both Graph and BiColGraph. This in fact follows from a more general observation. The definition of $\Omega(i \in I)$ in a presheaf $\mathsf{Set}^{I^{\mathrm{op}}}$ is the class of all subobjects of the presheaf $y(i)$ where y is the Yoneda embedding. Moreover, the class of subobjects of a presheaf form a Heyting algebra. The description of this Heyting algebra is given in the proof of [4, I.8 Prop. 5] and coincide what the ones given above in the case of Graph and BiColGraph.

A (Lawvere-Tierney) ***topology*** on a topos is a morphism $j : \Omega \to \Omega$ satisfying axioms (1)-(3) below. A topology j induces a ***closure*** operator on the subobjects: given $A_0 \rightarrowtail A$, then $\overline{A_0} \rightarrowtail A$ is defined by $\chi_{\overline{A_0}} := j \circ \chi_{A_0}$.

$$(1)\ j \circ \mathsf{True} = \mathsf{True}, \qquad (2)\ j \circ j = j, \qquad (3)\ j \circ \wedge = \wedge \circ (j \times j).$$

There are exactly 4 topologies on Graph [10]. For each of them, we describe the closure $\overline{G}$ of a subgraph $G \subseteq H$. The closure w.r.t. the *discrete topology*, $j = \mathrm{id}_\Omega$, adds nothing: $\overline{G} = G$. The closure w.r.t. the *closed topology for st*, $j = - \vee st$ [5, p. 197], adds all vertices: $\overline{G} = G \cup H(V)$. The closure w.r.t. the *double negation topology*, $j = \neg\neg$, adds all edges with source and target already in G: $\overline{G} = G \cup \big(H(E) \cap (V(G) \times V(G))\big)$. The closure w.r.t. the *trivial topology*, $j = \mathsf{True}_\Omega := \Omega \xrightarrow{!} 1 \xrightarrow{\mathsf{True}} \Omega$, adds everything: $\overline{G} = H$. Analogously, we obtain the next lemma.

Lemma 2. *There are 8 topologies in* BiColGraph*:*

1. *$j_1 = \mathrm{id}_\Omega$ the discrete topology,*
2. *j_2 is $\neg\neg$ on E and id on E',*
3. *j_3 is id on E and $\neg\neg$ on E',*
4. *$j_4 = \neg\neg$ (on E and on E'),*
5. *j_5 is $- \vee st$ on E and $- \vee s't'$ on E',*
6. *j_6 is $- \vee st$ on E and trivial on E',*
7. *j_7 is trivial on E and $- \vee s't'$ on E',*
8. *$j_8 = \mathsf{True}_\Omega$ is trivial (on E and on E').*

Proof. In BiColGraph, the terminal object 1 is $\circlearrowleft \cdot \circlearrowright$, and the image of $\mathsf{True} : 1 \to \Omega$ is $1 \circlearrowleft 1 \circlearrowright 1'$. Because of (1), a topology $j : \Omega \to \Omega$ on BiColGraph must leave the image of True untouched, i.e., j sends the vertex 1 and the edges 1 and $1'$ to themselves. For the vertex $0 \in \Omega(V)$, there are two choices.

The first choice is $j(0) = 0$. Then the edges $0, 0', s, s', t, t'$ have no other choice than to be sent to themselves, because their source and target are fixed. Only the edges st and $s't'$ remain to be mapped, each having two possible choices: for $j(st)$ either st or 1, and for $j(s't')$ either $s't'$ or $1'$. That gives us the 4 topologies j_1, j_2, j_3 and j_4.

The second choice is $j(0) = 1$. Then the edge 0 can be mapped to either st or 1. Similarly, the edge $0'$ can be mapped to either $s't'$ or $1'$. We will see that this determines the mapping of the rest of the edges. That gives us the remaining 4 topologies j_5, j_6, j_7 and j_8. Let us do the case of j_5. We have $j_5(0) = st$ and $j_5(0') = s't'$. Because j_5 is idempotent by (2), st and $s't'$ must then be fixed points of j_5, i.e., $j_5(st) = st$ and $j_5(s't') = s't'$. In general we have that $a \leqslant b$ in Ω implies $j_5(a) \leqslant j_5(b)$ because by (2): $j(a) = j(a \wedge b) = j(a) \wedge j(b)$. In our case, we have $0 < s, t < st$ and both 0 and st have the same image. Therefore, s and t must also have the same image, i.e., $j_5(s) = j_5(t) = st$. Similarly, $j_5(s') = j_5(t') = s't'$. This means that j_5 is the topology closed for st and closed for $s't'$. $\square$

Given a topology j on a topos, there is a notion of j-separated elements and the result that j-separated elements form a full subcategory which is a quasitopos [3, Thm. 10.1]. Let us recall the definitions needed. A subobject $A_0 \rightarrowtail A$ is said to be ***j-dense*** if $\overline{A_0} = A$ [4, p. 221]. An object B is called ***j-separated*** if for every j-dense subobject $m : A_0 \rightarrowtail A$ and every morphism $f : A_0 \to B$ there exists at most one factorisation $g : A \to B$ of f through m [4, p. 223].

$$\begin{tikzcd} A_0 \arrow[d, tail, "m"'] \arrow[dr, "f"] & \\ A \arrow[r, dotted, "g"'] & B \end{tikzcd}$$

In the case of Graph and of $j = \neg\neg$, having a $\neg\neg$-dense subgraph $A_0 \subseteq A$ means that A_0 contains all the vertices of A. Hence, a graph homomorphism $f : A_0 \to B$ fixes the image of all the vertices of A. The number of $g : A \to B$ factorising f then depends on how many choices each edge of A has inside B. Therefore, B is $\neg\neg$-separated if and only if B has no parallel edges, i.e., is a simple graph (with loops allowed). By [3, Thm. 10.1], we thus have that simple graphs form a quasitopos.

In $\mathsf{BiColGraph}$, we obtain similar results using the topologies described in Lemma 2. B is j_2-separated if it has no parallel blue edges and is j_3-separated if it has no parallel red edges. Both cases are symmetric, as we can swap the colours. Such graphs, for which one of the set of edges is simple, we call ***partially simple bicoloured graphs***. Using again that separated objects form a full subcategory which is a quasitopos [3, Thm. 10.1], we have the following.

Theorem 3. *Partially simple bicoloured graphs, i.e., bicoloured graphs where no parallel edges are allowed for only one edge colour, form a quasitopos.*

3 Conclusion

There are several directions for future work. We want to explore more topologies in other presheaf categories, such as simplicial sets; as well in other non-presheaf toposes. By considering the subcategories of the separated elements of these topologies, more potentially useful quasitoposes may arise.

This process of obtain new quasitoposes can also be done when starting from a quasitopos (instead of starting from a topos). Indeed, we recalled the definition of a topology on a topos, but there is also a slightly more general definition of a topology on a quasitopos [11, 41.1]. We used that definition to prove that the category of simple fuzzy graphs is a quasitopos [9, Lemma 53]. It might moreover be the case that after having considered one topology and having restricted ourselves to the separated elements, a new interesting topology might exist on the quasitopos so obtained, motivating a re-separation for obtaining a second quasitopos.

Finally, one may wonder if the internal logic of the quasitopos obtained via separation can be deduced from the internal logic of the (quasi)topos that we started with.

Acknowledgments

The author received funding from the Netherlands Organization for Scientific Research (NWO) under the Innovational Research Incentives Scheme Vidi (project. No. VI.Vidi.192.004).

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