Computational content of a generalized Kreisel-Putnam rule

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The Harrop rule (Harrop (1960)) also known as the Independence of Premise rule or the Kreisel-Putnam rule:

$$\frac{\neg C \to (A \vee B)}{(\neg C \to A) \vee (\neg C \to B)}\ \text{Harrop}$$

is an intriguing rule. It is an admissible but not a derivable rule of intuitionistic logic (Iemhoff (2001)), despite being proof-theoretically valid (Piecha et al. (2014)) in a variant of Dummett-Prawitz-style semantics (Prawitz (1971)). If we add it to the intuitionistic logic, we obtain the Kreisel-Putnam logic (Kreisel and Putnam (1957)), which is stronger than the intuitionistic logic yet still has the disjunction property (whenever $A \vee B$ is a theorem, either A or B is a theorem), previously thought to be a property specific to the intuitionistic logic. Furthermore, it is admissible in any intermediate logic (Prucnal (1979)).

Yet, its generalized version, which we call the Split rule:[1]

$$\frac{C \to (A \vee B)}{(C \to A) \vee (C \to B)}\ \text{Split}$$

is arguably even more interesting. If we add it to the intuitionistic logic, we obtain inquisitive intuitionistic logic (Punčochář (2016)), which has both the disjunctive property and the structural completeness property (enjoyed by classical logic: every admissible rule is derivable), again it can be shown to be proof-theoretically valid in a variant of Dummett-Prawitz-style semantics (Stafford (2021)), yet it is not closed under uniform substitution. Furthermore, it is admissible in any intermediate logic (Minari and Wronski (1988)) and it also makes a surprising appearance in domain logics (Abramsky (1991)) and we are confident that this list is not complete.

Despite its significance, the Split rule itself remains mostly unexplored, especially in terms of its proof-theoretic meaning and computational content (a recent exception to this is Condoluci and Manighetti (2018) examining the admissibility of the related Harrop rule from the computational view). In this paper, we fill this gap and propose a computational interpretation of the Split rule. We will achieve this by exploiting the Curry-Howard correspondence between formulas and types (also known as the propositions-as-types principle). First, we inspect the inferential behavior of the Split rule in the setting of a natural deduction system for the intuitionistic propositional logic. This will then guide our process of formulating an appropriate program that would capture the corresponding computational content of the typed Split rule. In other words, we

[1] Where C is a Harrop formula, also known as Rasiowa-Harrop formula (Rasiowa (1954)), i.e., a formula in which every disjunction occurs only within the antecedents of implications.

want to find an appropriate selector function (i.e., a noncanonical eliminatory operator) for the Split rule by considering its typed variant. Our investigation can be thus also reframed as an effort to answer the following questions: is the Split rule constructively valid in the style of BHK semantics? In other words, can we find a constructive function that would transform arbitrary proofs of the premise of the Split rule into proofs of its conclusion?

We propose two possible selectors S and FS corresponding to the two possible generalizations of the typed Split rule: the variant S is based on the selector for the typed disjunction elimination rule, and the other variant FS is based on the selector for the typed general implication elimination rule. Both variants are equivalent, but the latter requires the adoption of rules with higher-level assumptions, i.e., assumptions that depend on other assumptions.

The typed rule S takes the following form:

$$\dfrac{\begin{array}{c}[z:C]\\ c(z):A\vee B\end{array}\qquad\begin{array}{c}[x:C\to A]\\ d(x):D\end{array}\qquad\begin{array}{c}[y:C\to B]\\ e(y):D\end{array}}{\mathsf{S}(z.c,x.d,y.e):D}\,\mathrm{S}$$

with the computation rules $\mathsf{S}(z.\mathsf{i}(a(z)),x.d,y.e)=d(\lambda z.a(z)/x):D$ and $\mathsf{S}(z.\mathsf{j}(b(z)),x.d,y.e)=e(\lambda z.b(z)/y):D$. The rules FS takes the following form:

$$\dfrac{f:C\to(A\vee B)\qquad\begin{array}{c}\begin{bmatrix}[x:C]\\ y(x):A\end{bmatrix}\\ d(y):D\end{array}\qquad\begin{array}{c}\begin{bmatrix}[x:C]\\ w(x):B\end{bmatrix}\\ e(w):D\end{array}}{\mathsf{FS}(f,y.d,w.e):D}\,\mathrm{FS}$$

with the computation rules $\mathsf{FS}(\lambda(\mathsf{i}(a)),y.d,w.e)=d(a):D$ and $\mathsf{FS}(\lambda(\mathsf{j}(b)),y.d,w.e)=e(b):D$. Thus, the computational content of the S rule is expressed by the program S, or, if we allow higher-level assumptions (corresponding to function variables), by the higher-level program FS. Furthermore, we consider two additional variants S$'$ and FS$'$ formed by "mixing and matching" aspects of the rules S and FS.

With these selectors at hand, we can claim that the S rule is constructively valid. And since the S rule and the Split rule are interderivable, we can further claim that the Split rule is constructively valid as well.

Note that the FS rule has in comparison with the S rule a number of advantages: we do not have to reduce the original premise of the Split into a hypothetical derivation, we can just keep it as it is and treat the rule as an elimination-like rule for implication (in other words, the major premises of the Split rule and the FS rule are the same, which is not the case for the Split rule and the S rule). Furthermore, we do not need to introduce the auxiliary implication assumptions as in the S rule and instead handle the dependency between $A\vee B$ and C more directly via the notion of a higher-level assumption.

Finally, we show that extending intuitionistic propositional logic with the S rule preserves strong normalization, subject reduction, and the disjunction property.

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