COMPLETE REPRESENTATION BY PARTIAL FUNCTIONS FOR SIGNATURES CONTAINING ANTIDOMAIN RESTRICTION

BRETT MCLEAN

Abstract. We investigate notions of complete representation by partial functions, where the operations in the signature include antidomain restriction and may include composition, intersection, update, preferential union, domain, antidomain, and set difference. When the signature includes both antidomain restriction and intersection, the join-complete and the meet-complete representations coincide. Otherwise, for the signatures we consider, meet-complete is strictly stronger than join-complete. A necessary condition to be meet-completely representable is that the atoms are separating. For the signatures we consider, this condition is sufficient if and only if composition is not in the signature. For each of the signatures we consider, the class of (meet-)completely representable algebras is not axiomatisable by any existential-universal-existential first-order theory. For 14 expressively distinct signatures, we show, by giving an explicit representation, that the (meet-)completely representable algebras form a basic elementary class, axiomatisable by a universal-existential-universal first-order sentence. The signatures are those containing antidomain restriction and any of intersection, update, and preferential union and also those containing antidomain restriction, composition, and intersection and any of update, preferential union, domain, and antidomain.

Keywords: complete representation; partial function; antidomain restriction; finite first-order axiomatisation.

In [2], Jackson and Stokes investigate the axiomatisability of classes of algebras that are representable as (i.e. isomorphic to) an algebra of partial functions. Using a uniform method of representation, they give, for around 30 different signatures containing the *domain restriction* operation, either a finite equational or finite quasi-equational axiomatisation of the class of representable algebras. Only a handful of these classes had previously been axiomatised.

Here, we show that a similar uniform method of representation can be used to characterise many of the corresponding subclasses of completely representable algebras. A complete representation is one that turns any existing infima/suprema into intersections/unions. Specifically, we do this for signatures containing the operation called *minus* in [2] and which we call *antidomain restriction*; thus for about half of the signatures treated in [2]. Together with the results of [2], this gives us finite first-order axiomatisations of the classes of completely representable algebras for 14 expressively distinct signatures. Only a couple of complete representation classes had previously been axiomatised (for representation as partial functions) [3, 1].

We now give formal definitions and statements of our results.

Definition 1. Let σ be an algebraic signature whose symbols are a subset of $\{\rhd', ;, \wedge, [\ ], \sqcup, \mathrm{D}, \mathrm{A}\}$. An **algebra of partial functions** of the signature σ is an algebra of the signature σ whose elements are partial functions and with operations given by the set-theoretic operations on those partial functions described in the following.

Let X be the union of the domains and ranges of all the partial functions. We call X the **base**. The operations are defined as follows.

- The binary operation $\rhd'$ is **antidomain restriction**. It is the restriction of the second argument to elements *not* in the domain of the first; that is:
$$f \rhd' g \coloneqq \{(x, y) \in X^2 \mid x \notin \mathrm{dom}(f) \text{ and } (x, y) \in g\}.$$
- The binary operation ; is **composition** of partial functions:
$$f \mathbin{;} g = \{(x, z) \in X^2 \mid \exists y \in X (x, y) \in f \text{ and } (y, z) \in g\}.$$
- The binary operation $\wedge$ is **intersection**:
$$f \wedge g = \{(x, y) \in X^2 \mid (x, y) \in f \text{ and } (x, y) \in g\}.$$
- The binary operation [ ] is **update**:
$$f[g](x) = \begin{cases} f(x) & \text{if } f(x) \text{ defined but } g(x) \text{ undefined} \\ g(x) & \text{if } f(x) \text{ undefined and } g(x) \text{ defined} \\ \text{undefined} & \text{otherwise.} \end{cases}$$
- The binary operation $\sqcup$ is **preferential union**[1]:
$$(f \sqcup g)(x) = \begin{cases} f(x) & \text{if } f(x) \text{ defined} \\ g(x) & \text{if } f(x) \text{ undefined, but } g(x) \text{ defined} \\ \text{undefined} & \text{otherwise.} \end{cases}$$
- The unary **domain** operation D is the operation of taking the diagonal of the domain of a function:
$$\mathrm{D}(f) = \{(x, x) \in X^2 \mid x \in \mathrm{dom}(f)\}.$$
- The unary **antidomain** operation A is the operation of taking the diagonal of the antidomain of a function—those elements of X where the function is not defined:
$$\mathrm{A}(f) = \{(x, x) \in X^2 \mid x \in X \setminus \mathrm{dom}(f)\}.$$

Additionally, given that antidomain restriction is in the signature, the presence of the following are equivalent, respectively, to the presence of antidomain and the presence of intersection; hence we do not need to consider them independently.

- The constant 1 is the **identity** function on X:
$$1 = \{(x, x) \in X^2\}.$$
We have $1 = \mathrm{A}(\mathrm{A}(f))$ and $\mathrm{A}(f) = f \rhd' 1$.
- The binary operation $\setminus$ is **relative complement**:
$$f \setminus g = \{(x, y) \in X^2 \mid (x, y) \in f \text{ and } (x, y) \notin g\}.$$
We have $f \setminus g = (f \wedge g) \rhd' f$ and $f \wedge g = f \setminus (f \setminus g)$.

The list of operations we have given does not exhaust those that have been considered for partial functions, but does include many of the most commonly appearing operations. Notable exceptions are *range* and *range restriction*.

[1]This operation is also known as *override*.

Definition 2. Let $\mathfrak{A}$ be an algebra of one of the signatures specified by Definition 1. A **representation of $\mathfrak{A}$ by partial functions** is an isomorphism from $\mathfrak{A}$ to an algebra of partial functions of the same signature. If $\mathfrak{A}$ has a representation then we say it is **representable**.

The following theorem is stated for precisely the signatures we are interested in.

Theorem 3 (Jackson and Stokes [2])**.** *Let* $\{\rhd'\} \subseteq \sigma \subseteq \{\rhd', \wedge, [\ ], \sqcup\}$ *or* $\{\rhd', ;\} \subseteq \sigma \subseteq \{\rhd', ;, \wedge, [\ ], \sqcup, \mathrm{D}, \mathrm{A}\}$ *(see Figure 1). Then the class of σ-algebras representable by partial functions is a finitely based variety or a finitely based quasivariety.*

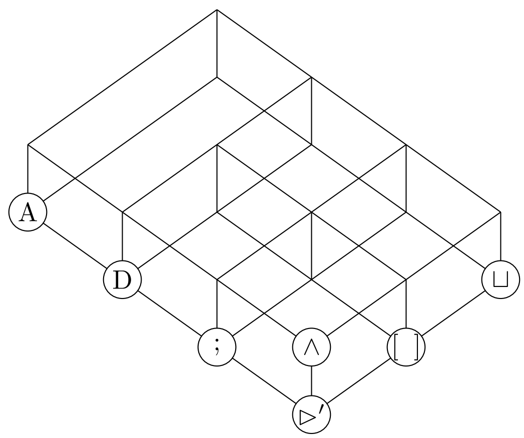

Figure 1. Hasse diagram of the signatures investigated and their relative expressiveness. Each vertex represents the signature containing the operations appearing below the vertex.

If an algebra of a signature containing $\rhd'$ is representable by partial functions, then it forms a poset when equipped with the relation $\leq$ defined by

$$a \leq b \iff a \rhd b = a.$$

Definition 4. A representation θ of a poset $\mathfrak{P}$ over the base X is **meet complete** if, for every *nonempty* subset S of $\mathfrak{P}$, if $\bigwedge S$ exists, then $\theta(\bigwedge S) = \bigcap \theta[S]$. It is **join complete** if, for every subset S of $\mathfrak{P}$, if $\bigvee S$ exists, then $\theta(\bigvee S) = \bigcup \theta[S]$.

Proposition 5. *Let σ be a signature including $\{\rhd', \wedge\}$. Let $\mathfrak{A}$ be a σ-algebra and θ be a representation of $\mathfrak{A}$ by partial functions. Then θ is meet complete if and only if it is join complete.*

Thus for these signatures we may drop the specifiers 'meet' and 'join' and just speak of 'complete' representations. Our main result is the following.

Theorem 6. *Let* $\{\rhd'\} \subseteq \sigma \subseteq \{\rhd', \wedge, [\ ], \sqcup\}$ *or* $\{\rhd', ;, \wedge\} \subseteq \sigma \subseteq \{\rhd', ;, \wedge, [\ ], \sqcup, \mathrm{D}, \mathrm{A}\}$*. Then the class of σ-algebras that are (meet-)completely representable by partial functions is a basic elementary class, axiomatisable by a universal-existential-universal first-order sentence.*

The central ingredient in the proof is the following representation theorem, where $\sim$ denotes the 'have the same domain' relation defined by

$$a \sim b \iff (a \rhd b = b \text{ and } b \rhd a = a).$$

The representations are Cayley-style representations but also have a certain similarity to the Birkhoff–Stone representation, for the representations use atoms for their base, and atoms correspond to principal ultrafilters in Boolean algebras.

Proposition 7. *Let* $\{\rhd', ;\} \subseteq \sigma \subseteq \{\rhd', ;, \wedge, [\ \ ], \sqcup, \mathrm{D}, \mathrm{A}\}$*, and let* $\mathfrak{A}$ *be a* σ*-algebra. Suppose* $\mathfrak{A}$ *is representable by partial functions and the atoms* $\mathrm{At}(\mathfrak{A})$ *of* $\mathfrak{A}$ *are separating.*[2] *If* $\mathrm{D}, \mathrm{A} \notin \sigma$*, for each* $a \in \mathfrak{A}$*, let* $\theta(a)$ *be the following partial function on the disjoint union* $\mathrm{At}(\mathfrak{A}) \amalg \mathrm{At}(\mathfrak{A})/{\sim}$*. For* $x \in \mathrm{At}(\mathfrak{A})$

$$\theta(a)(x) = \begin{cases} x \mathbin{;} a & \text{if } x \mathbin{;} a \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

and

$$\theta(a)(x/{\sim}) = \begin{cases} x \rhd a & \text{if } x \rhd a \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

If $\mathrm{D} \in \sigma$ *or* $\mathrm{A} \in \sigma$*, for each* $a \in \mathfrak{A}$*, let* θ *be only the first component of the partial function just defined, so* $\theta(a)$ *is a partial function on* $\mathrm{At}(\mathfrak{A})$*. Then* θ *is a representation of* $\mathfrak{A}$ *by partial functions, with base either* $\mathrm{At}(\mathfrak{A}) \amalg \mathrm{At}(\mathfrak{A})/{\sim}$ *or* $\mathrm{At}(\mathfrak{A})$ *as appropriate.*

Further,

(1) if, in $\mathfrak{A}$*, composition is completely left-distributive over joins, then* θ *is join complete;*
(2) if, in $\mathfrak{A}$*, composition is completely left-distributive over meets, then* θ *is meet complete.*

References

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Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Ghent, Belgium

Email address: brett.mclean@ugent.be

[2]We say that the **atoms are separating** if whenever $a \not\leq b \in \mathfrak{A}$ then there exists an atom c of $\mathfrak{A}$ with $c \leq a$ and $c \not\leq b$.