DECIDABILITY OF NEIGHBORHOOD PRODUCTS OF MODAL LOGICS

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The products of two modal logics (see [10, 2]) are in many cases undecidable. In [3] it was proved that under some weak restrictions the product of transitive logics is undecidable. In particular any product of logics $\mathsf{K4}, \mathsf{D4}, \mathsf{S4}$ is undecidable. We show that for neighborhood products of these logics the situation is different. In many cases the neighborhood product of modal logics is decidable and sometimes even is in PSPACE.

A *neighborhood frame* (or an n-frame) is a pair $\mathcal{X} = (X, \tau)$, where X is a nonempty set and $\tau : X \to 2^{2^X}$ is the *neighborhood function* of $\mathcal{X}$. Sets from $\tau(x)$ are called *neighborhoods* of point x. *A neighborhood model* (n-model) is a pair $M = (\mathcal{X}, V)$, where $\mathcal{X} = (X, \tau)$ and V is a valuation on X, i.e. a function from the set of all propositional variables to set 2^X. The truth relation $\models$ on n-models for logical connectives is defined in the classical way and for modalities the definition is the following:

$$M, x \models \Box A \iff \exists U \forall y (y \in U \in \tau(x) \Rightarrow M, y \models A).$$

A formula A is *valid* on an n-frame $\mathcal{X}$ if it is true at all points on all models based on $\mathcal{X}$ (Notation: $\mathcal{X} \models A$). For a class of n-frames $\mathcal{C}$ formula A is *valid* on $\mathcal{C}$ if $\forall \mathcal{X} \in \mathcal{C}(\mathcal{X} \models A)$. The set of all formulas that are valid on a given frame (a class of frames) is called the logic of this frame (class of frames). Notations: $Log(F)$ and $Log(\mathcal{C})$ respectively.

Any logic of a Kripke frame is normal, i.e. it contains axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and is closed under the necessitation rule $(\frac{A}{\Box A})$. Whereas logic of an n-frame is not necessarily normal. For it to be normal $\tau(x)$ has to be a filter for any $x \in \mathcal{X}$. *Filter* is a nonempty set of sets closed under intersections and supersets. From now on we will assume that all n-frames are like this.

N-frames is a more general notion then a Kripke frame. Given a Kripke frame $F = (W, R)$ we can construct an n-frame $\mathcal{N}(F) = (W, \tau_R)$, such that $\tau_R(x) = \{U \subseteq W \mid R(x) \subseteq U\}$. Then the validity of formulas will be preserved by this operation.

One of the natural construction that "increase" dimension is the product. For two Kripke frames $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$ their product is a Kripke frame with 2 relations $F_1 \times F_2 = (W_1 \times W_2, R_1^h, R_2^v)$, where

$$(x, y) R_1^h (z, t) \iff x R_1 z \ \&\ y = t;$$
$$(x, y) R_2^v (z, t) \iff x = z \ \&\ y R_2 t.$$

In a similar way we can define the product of two n-frames. Let $\mathcal{X}_1 = (X_1, \tau_1)$ and $\mathcal{X}_2 = (X_2, \tau_2)$ be two n-frames. Their product is an n-frame with 2 neighborhood functions:

$$\mathcal{X}_1 \times \mathcal{X}_2 = (X_1 \times X_2, \tau_1^h, \tau_2^v), \text{ where}$$
$$\tau_1^h(x_1, x_2) = \{U \subseteq X_1 \times X_2 \mid \exists V (V \in \tau_1(x_1) \ \&\ V \times \{x_2\} \subseteq U)\};$$
$$\tau_2^v(x_1, x_2) = \{U \subseteq X_1 \times X_2 \mid \exists V (V \in \tau_2(x_2) \ \&\ \{x_1\} \times V \subseteq U)\}.$$

In frames with several relations (neighborhood functions) we use modalities with corresponding sub indexes.

For two unimodal logics L_1 and L_2, we define the *n-product* of them as a logic with two modalities:

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = Log\big(\{\mathcal{X}_1 \times \mathcal{X}_2 \mid \mathcal{X}_1 \models \mathsf{L}_1 \ \&\ \mathcal{X}_2 \models \mathsf{L}_2\}\big).$$

The fusion of L_1 and L_2 is the minimal 2-modal logic $\mathsf{L}_1 * \mathsf{L}_2$ axiomatized with axioms of L_1 rewritten with $\Box_1$ and axioms of L_2 rewritten with $\Box_2$.

Topological semantics is a particular case of neighborhood semantics for extensions of $\mathsf{S4}$. For a topological space $\mathfrak{X} = (X, T)$ we can define neighborhood function

$$\tau(x) = \{U \mid \exists U' \in T(x \in U' \subseteq U)\}.$$

The truth relation on $\mathfrak{X}$ and on $\mathcal{X} = (X, \tau)$ will be the same. First paper where topological product was defined was [11]. The authors showed that $\mathsf{S4} \times_n \mathsf{S4} = \mathsf{S4} * \mathsf{S4}$.

The *derivational* topological semantics (d-semantics) is also a particular case of neighborhood semantics. For a topological space $\mathfrak{X} = (X, T)$ we can define neighborhood function

$$\tau_d(x) = \{U \mid \exists U' \in T(x \in U' \ \&\ U' \setminus \{x\} \subseteq U)\}.$$

The truth relation on $\mathfrak{X}$ with d-semantics and on $\mathcal{X} = (X, \tau_d)$ will also be the same. The results on the d-products of modal logics (the d-logic of the products of topological spaces) can be found in [5]. So the n-frames for extensions of $\mathsf{K4}$ is basically the d-semantics on topological spaces.

Let us put all known completeness results from [7, 11] into a table:

L_1	L_2	$\mathsf{L}_1 \times_n \mathsf{L}_2$
$\mathsf{S4}$	$\mathsf{S4}$	$\mathsf{S4} * \mathsf{S4}$
HTC-logic $+\Diamond\top$	HTC logic $+\Diamond\top$	$\mathsf{L}_1 * \mathsf{L}_2$
HTC-logic	HTC-logic	$\mathsf{L}_1 * \mathsf{L}_2 + \Delta$
HTC-logic $+\Diamond\top$	HTC-logic	$\mathsf{L}_1 * \mathsf{L}_2 + \Delta_2$

Notation:

$$\Delta_1 = \{\phi \to \Box_1\phi \mid \phi \text{ is closed and } \Box_1\text{-free}\},$$
$$\Delta_2 = \{\psi \to \Box_2\psi \mid \psi \text{ is closed and } \Box_2\text{-free}\},$$
$$\Delta = \Delta_1 \cup \Delta_2.$$

A logic L is called *HTC-logic* (Horn preTransitive Closed logic) if it can be axiomatized by closed formulas and formulas of the type $\Box p \to \Box^n p$ or $p \to \Box^n p$.

Logics $\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}, \mathsf{S4}$ are all HTC-logics.

Theorem 1. *Let* $\mathsf{L}_1, \mathsf{L}_2 \in \{\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}, \mathsf{S4}\}$ *then the n-product* $\mathsf{L}_1 \times_n \mathsf{L}_2$ *has fmp and decidable.*

If both logic are serial then their n-product equals to the fusion and fusion of decidable logics is decidable (see. [2]). The fusion even preserves complexity so in this case the product is PSPACE-complete.

In other cases we need to show that adding Δ and Δ_2 will keep the fmp and the decidability. This is the main result of this paper.

Further work can include studding the exact complexity of these logics. And also the decidability and complexity of other products of modal logics.

References

[1] D. Gabbay and V. Shehtman. Products of modal logics. Part I. *Journal of the IGPL*, 6:73–146, 1998.

[2] D.M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics : theory and applications*, volume 148 of *Studies in logic and the foundations of mathematics*. Elsevier, 2003.

[3] D. Gabelaia, A. Kurucz, F. Wolter, and M. Zakharyaschev. Products of 'transitive' modal logics. *The Journal of Symbolic Logic*, 70(3):993–1021, 2005.

[4] P. Kremer. The topological product of S4 and S5. *Unpublished*, 2011.

[5] A. Kudinov. D-logic of product of rational numbers. In *Procedings of Information Technology and Systems*, pages 95–99, Moscow, 2013.
[6] A. Kudinov. Neighborhood-kripke product of modal logics. *Topology, Algebra, and Categories in Logic (TACL'17)*, 2017.
[7] A. Kudinov. On neighbourhood product of some horn axiomatizable logics. *Logic Journal of the IGPL*, 26(3):316–338, 2018.
[8] A. Kudinov. Neighbourhood completeness for quantified pretransitive modal logics. arXiv, 2021.
[9] K. Sano. Axiomatizing hybrid products of monotone neighborhood frames. *Electr. Notes Theor. Comput. Sci.*, 273:51–67, 2011.
[10] V. Shehtman. Two-dimensional modal logic. *Mathematical Notices of USSR Academy of Science*, 23:417–424, 1978. (Translated from Russian).
[11] J. van Benthem, G. Bezhanishvili, B. Cate, and D. Sarenac. Multimodal logics of products of topologies. *Studia Logica*, 84:369–392, 2006.

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