Implicational Fragments of Some Subintuitionistic Logics

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In a series of papers [3, 4, 5, 6] we discussed a number of subintuitionistic logics between WF and IPC. In this paper we investigate the implicational fragments of these logics. We denote a fragment by listing the connectives between square brackets, so $[\to]$ is the fragment consisting of formulas that only contain the connective $\to$. For any subintuitionistic logic $\mathcal{L}$ we define $\mathcal{L}_{[\to]}$ as the logic consisting of the formulas with $\to$ only provable in $\mathcal{L}$. Mainly, we try to establish an axiomatization of $\mathcal{L}_{[\to]}$. We have an interest in $\mathcal{L}_{[\to,\wedge]}$, the $[\to,\wedge]$-fragment of $\mathcal{L}$, as well. This fragment is in IPC closely related to the $[\to]$-fragment [7], and sometimes easier to describe. This is research in progress.

The language of subintuitionistic logics, is the same language as that of IPC. It contains the connectives $\vee, \wedge, \to$ and the propositional constant $\bot$. Moreover, it contains a denumerable set of propositional variables.

Definition 1. WF *is the logic given by the following axioms and rules,*

1. $A \to A \vee B$
2. $B \to A \vee B$
3. $A \to A$
4. $A \wedge B \to A$
5. $A \wedge B \to B$
6. $\frac{A \quad A\to B}{B}$
7. $\frac{A\to B \quad A\to C}{A\to B\wedge C}$
8. $\frac{A\to C \quad B\to C}{A\vee B\to C}$
9. $\frac{A\to B \quad B\to C}{A\to C}$
10. $\frac{A}{B\to A}$
11. $\frac{A\leftrightarrow B \quad C\leftrightarrow D}{(A\to C)\leftrightarrow(B\to D)}$
12. $\frac{A \quad B}{A\wedge B}$
13. $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
14. $\bot \to A$

Definition 2. *A triple $F = \langle W, g, NB\rangle$ is called an* ***NB-Neighborhood Frame*** *of subintuitionistic logic if W is a non-empty set, g is an element of W and NB is a neighborhood function from W into $P((P(W))^2)$ such that:*

1. $\forall w \in W,\ \forall X, Y \in P(W),\ (X \subseteq Y \ \Rightarrow\ (X, Y) \in NB(w))$;
2. $NB(g) = \{(X, Y) \in (P(W))^2 \mid X \subseteq Y\}$ *(g is called* ***omniscient****).*

Theorem 1. [12] *The logic* WF *is sound and strongly complete with respect to the class of NB-Neighborhood frames.*

To the system WF we add the rule N to obtain the logic $\mathsf{WF_N}$:

$$\frac{A \to B \vee C \quad C \to A \vee D \quad A \wedge C \wedge D \to B \quad A \wedge C \wedge B \to D}{(A \to B) \leftrightarrow (C \to D)} \qquad \mathsf{N}$$

The logic $\mathsf{WF_N}$ is complete with respect to the standard unary N-Neighborhood frames [5]. Frames as defined e.g. in [10]. In addition we consider the following axiom schemas and rules:

$$(A \to B) \wedge (B \to C) \to (A \to C) \qquad \mathsf{I}$$

$$(A \to B) \wedge (A \to C) \to (A \to B \wedge C) \qquad \mathsf{C}$$

$$(A \to C) \wedge (B \to C) \to (A \vee B \to C) \qquad \mathsf{D}$$

$$(A \to B \wedge C) \to (A \to B) \wedge (A \to C) \qquad \widehat{\mathsf{C}}$$

$$(A \to B) \to (C \wedge A \to C \wedge B) \qquad \mathsf{C_W}$$

$$(A \vee B \to C) \to (A \to C) \wedge (B \to C) \qquad \widehat{\mathsf{D}}$$

$$(A \to B) \leftrightarrow (A \to A \wedge B) \qquad \mathsf{N_b}$$

$$(A \wedge B \to C) \leftrightarrow (A \wedge B \to A \wedge C) \qquad \mathsf{N_c}$$

$$\frac{C \to A \vee D \qquad A \wedge C \wedge B \to D}{(A \to B) \to (C \to D)} \qquad \mathsf{N_2}$$

If $\Gamma \subseteq \{\mathsf{I}, \mathsf{C}, \mathsf{D}, \widehat{\mathsf{C}}, \mathsf{C_W}, \widehat{\mathsf{D}}, \mathsf{N_b}, \mathsf{N_c}, \mathsf{N_2}\}$, we will write $\mathsf{WF}\Gamma$ for the logic obtaines from WF by adding to WF the schemas and rules in Γ as new axioms and rules.

Proposition 1. *The rules 6, 9, 10, 11 and axiom 3 of Definition 1 axiomatize the system* $\mathsf{WF}_{[\to]}$.

Proposition 2.
$\mathsf{WF}_{[\to]} + \frac{A \to B}{(C \to A) \to (C \to B)}$ $(\mathsf{I_L}) = \mathsf{WF}\widehat{\mathsf{C}}_{[\to]}$.
$\mathsf{WF}_{[\to]} + \frac{A \to B}{(B \to C) \to (A \to C)}$ $(\mathsf{I_R}) = \mathsf{WF}\widehat{\mathsf{D}}_{[\to]}$.

Proposition 3. $\mathsf{WF}\widehat{\mathsf{C}}\widehat{\mathsf{D}}_{[\to]} = \mathsf{WF}_{[\to]} + \mathsf{I_L} + \mathsf{I_R}$.

Conjecture 1. The aximatization of $\mathsf{WF}_{[\to]}$ and $\mathsf{WF}_{\mathsf{N}[\to]}$ are the same.

Conjecture 2. The aximatization of $\mathsf{WF}\widehat{\mathsf{C}}\widehat{\mathsf{D}}_{[\to]}$, $\mathsf{WFI}_{[\to]}$ and $\mathsf{WF}_{\mathsf{N_2}[\to]}$ are the same.

In this article, we will also focus on fragments of subintuitionistic logics that contain $[\to, \wedge]$ and not $\vee$. We will prove the following propositions for these fragments.

Proposition 4. *The axioms 3, 4, 5 and rules 6, 7, 9, 10, 11 and 12 of Definition 1 axiomatize the system* $\mathsf{WF}_{[\to,\wedge]}$.

Proposition 5.
$\mathsf{WF}_{[\to,\wedge]} + \mathsf{C}_{[\to,\wedge]} = \mathsf{WFC}_{[\to,\wedge]}$.
$\mathsf{WF}_{[\to,\wedge]} + \mathsf{C_W}_{[\to,\wedge]} = \mathsf{WFC_W}_{[\to,\wedge]}$.
$\mathsf{WF}_{[\to,\wedge]} + \mathsf{N_a}_{[\to,\wedge]} = \mathsf{WFN_a}_{[\to,\wedge]}$.
$\mathsf{WF}_{[\to,\wedge]} + \mathsf{N_c}_{[\to,\wedge]} = \mathsf{WFN_c}_{[\to,\wedge]}$.

Corollary 1.
$\mathsf{WF}_{[\to,\wedge]} \neq \mathsf{WF}_{\mathsf{N}[\to,\wedge]}$.
$\mathsf{WF}\widehat{\mathsf{C}}\widehat{\mathsf{D}}_{[\to,\wedge]} \neq \mathsf{WF}_{\mathsf{N_2}[\to,\wedge]}$.

Definition 3. *A* ***rooted subintuitionistic Kripke frame*** *is a triple $\langle W, g, R\rangle$. R is a binary relation on W; $g \in W$, the* ***root*** *is* ***omniscient****, i.e. gRw for each $w \in W$.*

The logic F is the smallest set of formulas closed under instances of WF, C, D and I.

Theorem 2. [2, 11] *The logic* F *is sound and strongly complete with respect to the class of rooted subintuitionistic Kripke frames.*

Proposition 6. *Let $\bar{A}_n \to B$ stand for B if $n = 0$, for $A_1 \to B$ if $n = 1$, and $A_1 \to (A_2 \to ... \to (A_n \to B)...)$ if $n \geq 2$. The axiomatization of $(\mathsf{F}_{[\to]})$ is as follows:*

1. $A \to A$
2. $\dfrac{A \quad A \to B}{B}$
4. $\dfrac{A}{B \to A}$
5. $\dfrac{\bar{A}_n \to (B \to C) \qquad \bar{A}_n \to (C \to D)}{\bar{A}_n \to (B \to D)}, \qquad n \geq 0$

Completeness of the Hilbert system for $\mathsf{F}_{[\to]}$ is due to K. Došen [8] and here we give a new proof.

To the system F we add the following axiom schemas T, R, P and $\mathsf{P_T}$ to obtain the logics FT, FR, FP and $\mathsf{FP_T}$ respectively:

$$(A \to B) \to (C \to (A \to B)) \qquad \mathsf{T}$$

$A \wedge (A \to B) \to B$ R

$p \to (\top \to p)$ (*p a propositional letter*) P

$A \to (B \to A)$ $\mathsf{P_T}$

In [4] we proved that if the scheme $\bot \to A$ is ignored in F, then F and FP prove the same schemes, i.e. the schemes of $\mathsf{F}_{[\to,\wedge,\vee]}$ and $\mathsf{FP}_{[\to,\wedge,\vee]}$ are the same. It follows that:

Proposition 7. *The schemes of* $\mathsf{FP}_{[\to]}$ *are the schemes of* $\mathsf{F}_{[\to]}$.

Visser's logic BPC is in our terminology the same as FTP= $\mathsf{FP_T}$ as regards theorems. K. Kikuchi in [9] introduced a system which charecterizes the implicational fragment of BPC as follows:

$A \to A$ $\qquad$ $A \to (B \to A)$

$(\bar{A} \to (B \to \gamma)) \to ((\bar{A} \to (C \to B)) \to (\bar{A} \to (C \to \gamma)))$

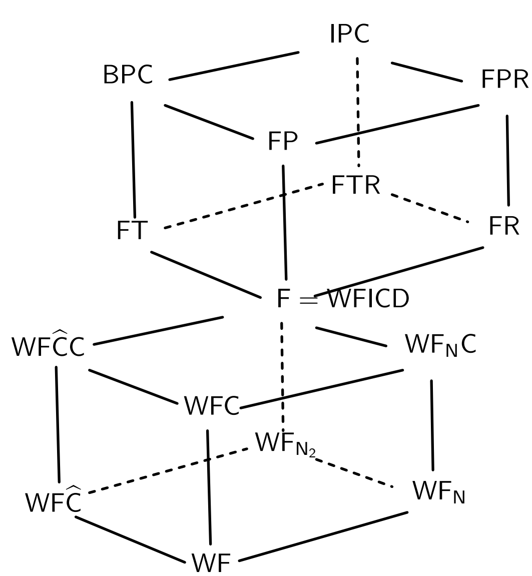
$\frac{A \quad A \to B}{B}$

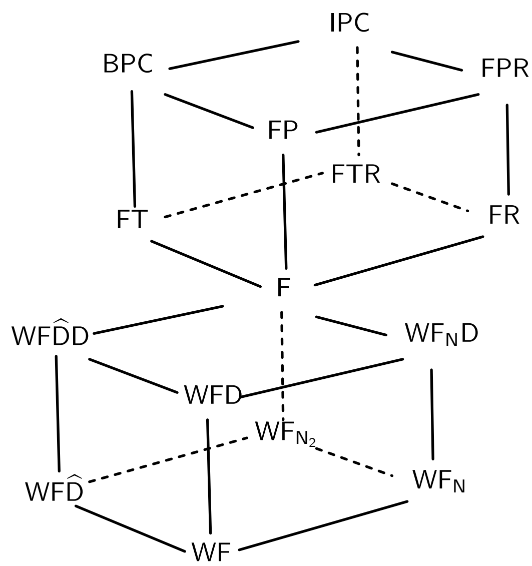
Proposition 8.

$\mathsf{F}_{[\to]} + \mathsf{T} = \mathsf{FT}_{[\to]}$.

$\mathsf{F}_{[\to]} + \mathsf{P_T} = \mathsf{FP}_{\mathsf{T}[\to]}$.

Generally, in this paper we will mainly investigate fragments of the subintuitionistic logics shown in the following picture.





The relations between the logics in the two bottom cubes is mostly yet unclear.

Finally, we proved the following Theorem:

Theorem 3. *None of the described subintuitionistic logics L below* IPC *have a locally finite fragment $L_{[\to]}$.*

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