Logic of Quantifier Shifts

Rosalie Iemhoff
Utrecht University

Raheleh Jalali
Utrecht University and
Czech Academy of Sciences

Skolemization is a method to remove strong quantifiers (i.e., positive occurrences of the universal quantifier and negative occurrences of the existential quantifier) from a first-order formula, and replace them with fresh function symbols. It is a well-known fact that Skolemization is sound and complete with respect to the classical predicate logic, CQC, while it is not the case for the intuitionistic predicate logic, IQC. Several studies have been done on the Skolemization in intermediate logics, including introducing alternative methods [1, 2, 3]. To explain more, let us first present the following formulas, that we call the *quantifier shifts*:

1. (Constant Domain) $\forall x(A(x) \vee B) \to \forall x A(x) \vee B$ (CD)
2. (Quantifier Switch) $(\forall x A(x) \to B) \to \exists x(A(x) \to B)$ (SW)
3. (Existential Distribution) $(B \to \exists x A(x)) \to \exists x(B \to A(x))$ (ED)

where $A(x)$ and B are formulas in the first-order language and the variable x is not free in B. None of these formulas are provable in IQC. However, in CQC, both these formulas and their converses are provable. One may suspect that the failure of the quantifier shifts is the reason why Skolemization fails in IQC. Therefore, it is natural to ask what happens if we add the quantifier shifts to IQC. Does the resulting logic have Skolemization? If not, for which class of formulas does the Skolemization hold? These questions build the motivation of the present research study that focuses first on investigating the *logic of quantifier shifts* and then its Skolemization. This talk is devoted to the first part of the study.

Denote the logic $\mathsf{IQC} + \{\mathrm{CD}, \mathrm{SW}, \mathrm{ED}\}$ by QFS. In the following, we will investigate the properties of this logic and its *fragments*, state the main results of this ongoing research, and sketch some of the proofs.

Definition 1 (Kripke frames and models)**.** [4, Chapter 14] A *Kripke frame* for IQC is a triple (W, R, D), where $W \neq \varnothing$ is a set of worlds, R is a binary reflexive and transitive relation over W, and D is a function assigning to each $w \in W$ a non-empty set $D(w)$, called the *domain* of w, such that if wRw' then $D(w) \subseteq D(w')$. A *Kripke model* for IQC is a quadruple (W, R, D, V) where (W, R, D) is a Kripke frame and V is a valuation function in its usual sense. A formula A is defined to be *valid* in a frame F, denoted by $F \vDash A$, and valid in a model M, denoted by $M \vDash A$, as usual. A Kripke frame is called *linear* when for any $w, w' \in W$ either wRw' or $w'Rw$. We call a Kripke frame *constant domain* when for any $w, w' \in W$, we have $D(w) = D(w')$.

First, let us observe the following easy facts that separate the fragments of QFS that we are interested in:

$$\mathsf{IQC} + \{\mathrm{CD}, \mathrm{SW}\} \nvdash \mathrm{ED} \quad \text{and} \quad \mathsf{IQC} + \{\mathrm{CD}, \mathrm{ED}\} \nvdash \mathrm{SW}.$$

Having this observation, we know that the logics QFS, $\mathsf{IQC} + \{\mathrm{CD}, \mathrm{SW}\}$, and $\mathsf{IQC} + \{\mathrm{CD}, \mathrm{ED}\}$ are all distinct. The following definition introduces rich classes of frames for these three logics.

Definition 2. Consider the following class $\mathcal{F}$ of Kripke frames:

1. Linear, constant domain, finite number of worlds (with finite/infinite domains)

2. Linear, constant domain, infinite number of worlds with finite domains.

3. Constant domain, and $D(w)$ has exactly one element, for any $w \in W$.

Then, define $\mathcal{F}_{\mathrm{SW}}$ (resp. $\mathcal{F}_{\mathrm{ED}}$) by adding all "linear, constant domain and conversely well-founded (well-founded)" frames to $\mathcal{F}$.

Theorem 3. *For any Kripke frame F:*

1. *$F \vDash \mathsf{QFS}$ if and only if $F \in \mathcal{F}$.*

2. *$F \vDash \mathsf{IQC} + \{\mathrm{CD}, \mathrm{SW}\}$ if and only if $F \in \mathcal{F}_{\mathrm{SW}}$.*

3. *$F \vDash \mathsf{IQC} + \{\mathrm{CD}, \mathrm{ED}\}$ if and only if $F \in \mathcal{F}_{\mathrm{ED}}$.*

One may wonder, why in the frame characterization, we always include the axiom CD. The reason simply is that if $F \vDash \mathsf{IQC} + \{\mathrm{SW}\}$ or $F \vDash \mathsf{IQC} + \{\mathrm{ED}\}$, then F must be constant domain and hence adding the axiom CD does not change the frame validity. However, note that it does not mean that the axiom scheme CD is provable in the mentioned logics.

Definition 4. The logic L is called *complete* with respect to the class $\mathcal{C}$ of Kripke frames when

$$L \vdash \varphi \quad \text{if and only if} \quad \mathcal{C} \vDash \varphi,$$

for any formula φ. The logic L is called *frame-complete* if there exists a class $\mathcal{C}$ of Kripke frames such that L is complete with respect to $\mathcal{C}$.

Theorem 5. *The logics* QFS*,* $\mathsf{IQC} + \{\mathrm{CD}, \mathrm{SW}\}$*, and* $\mathsf{IQC} + \{\mathrm{CD}, \mathrm{ED}\}$ *are all frame-incomplete.*

Proof. Let us sketch the proof for the case of QFS. To show that QFS is frame-incomplete, we have to prove that for any class $\mathcal{C}$ of Kripke frames for QFS, there exists a formula φ such that $\mathcal{C} \vDash \varphi$ but $\mathsf{QFS} \nvdash \varphi$. We claim that taking an instance of $\varphi = \mathrm{Lin} \vee \mathrm{OEP}$ works, where

$$\mathrm{Lin} := (C \to D) \vee (D \to C) \quad \text{and} \quad \mathrm{OEP} := \exists x A(x) \to \forall x A(x)$$

are the Linearity and One Element Principle schemes. To see why, we show that:

1. If a frame F validates QFS, then it is constant domain. Moreover, F is either linear or its domain is just a singleton. This means that F validates all instances of the axiom scheme $\mathrm{Lin} \vee \mathrm{OEP}$.

2. It is easy to see that there is an instance of the axiom scheme $\mathrm{Lin} \vee \mathrm{OEP}$ such that $\mathsf{QFS} \nvdash \mathrm{Lin} \vee \mathrm{OEP}$.

These two points together prove that QFS is frame-incomplete. □

Finally, as the last word in this extended abstract, let us recall the *propositional logic* of a first-order theory T, denoted by $\mathrm{PL}(L)$, as the set of all propositional formulas ϕ such that for any first-order substitution σ we have $T \vdash \sigma(\phi)$.

Theorem 6. $\mathrm{PL}(\mathsf{QFS}) = \mathsf{IPC}$.

This theorem intuitively states that, as expected, the quantifier shift formulas have no propositional content and hence adding them to IQC do not change the intuitionistic propositional logical base.

References

[1] Baaz, Matthias, and Rosalie Iemhoff. "Skolemization in intermediate logics with the finite model property." Logic Journal of the IGPL 24.3 (2016): 224-237.

[2] Iemhoff, Rosalie. "The eskolemization of universal quantifiers." Annals of Pure and Applied Logic 162.3 (2010): 201-212.

[3] Iemhoff, Rosalie. "The Skolemization of prenex formulas in intermediate logics." Indagationes Mathematicae 30.3 (2019): 470-491.

[4] Mints, Grigori. "A short introduction to intuitionistic logic." Springer Science & Business Media, 2000.