THE GOLDBLATT TRANSLATION REVISITED

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Ortholattices are algebras of type $(B, \wedge, ^{\perp}, 0, 1)$, satisfying all the axioms of Boolean algebras except possibly distributivity. These include various natural classes of algebras, such as the orthomodular lattices, studied classically by Birkhoff and Von Neumann, having close connections to an approach to quantum mechanics. They also have a logical counterpart:

Definition 1. *Let $\mathcal{L}_O = \{\wedge, ^{\perp}, 0, 1\}$ be the language of ortholattices. Let $\vdash$, be a binary relation between formulas of this language. Then we say that $\vdash$ is an orthologic if it is closed under uniform substitution, and satisfies the following axioms, for all $\phi, \psi, \chi \in \mathcal{L}_O$:*

(1) $\phi \wedge \psi \vdash \phi$; $\phi \wedge \psi \vdash \psi$
(2) $\phi \vdash \phi^{\perp\perp}$; $\phi^{\perp\perp} \vdash \phi$
(3) $\phi \wedge \neg\phi \vdash 0$ *and* $0 \vdash \psi$;
(4) $1 \vdash \phi \vee \neg\phi$ *and* $\psi \vdash 1$;
(5) *If* $\phi \vdash \psi$ *and* $\phi \vdash \chi$, *then* $\phi \vdash \psi \wedge \chi$
(6) *If* $\phi \vdash \psi$ *and* $\psi \vdash \chi$ *then* $\phi \vdash \chi$
(7) *If* $\phi \vdash \psi$ *then* $\psi^{\perp} \vdash \phi^{\perp}$

Given Γ a set of formulas, we write $\Gamma \vdash \phi$ to mean that there is a finite set $\Gamma_0 \subseteq \Gamma$ such that $\bigwedge \Gamma_0 \vdash \phi$.

Goldblatt [10] introduced orthologics (with a slightly different, but equivalent, axiomatisation), proved basic results about them including Kripke completeness and the finite model property, and provided a translation between Orthologic and the **KTB**-modal logic system. The latter is axiomatised over the basic modal logic **K** by adding the axioms

$$p \to \Box\Diamond p \text{ and } \Box p \to p.$$

The *Goldblatt translation* between $\mathcal{O}$ the minimal orthologic, and **KTB**, is as follows:

- $G(0) = 0$ and $G(1) = 1$.
- For $p \in \mathsf{Prop}$, $G(p) = \Box\Diamond p$;
- $G(\psi \wedge \phi) = G(\psi) \wedge G(\phi)$.
- $G(\phi^{\perp}) = \Box\neg G(\phi)$

Miyazaki [12] developed the theory of this translation in parallel with the classical Godel-McKinsey-Tarski (GMT) translation between intuitionistic and **S4** modal logic. There, he proved the existence of "modal companions" for each extension of orthologic with the finite model property, as well as the fact that each Kripke complete **KTB** logic is the modal companion of some orthologic. More general questions concerning the relationship between these logics were left open.

In this paper, based on a recent masters thesis [1], we analyse the Goldblatt translation through the combined perspective of algebraic logic, category theory and duality theory.

Date: 25 January 2022.

Our general approach seeks to go beyong conservativity and faithfulness of a translation, and instead presents two groups of criteria for judging what counts as a "good translation":

(1) *Generality*: The same translation schema also adequately translates extensions of the logic in question;
(2) *Strongness*: The translation preserves and reflects several natural properties, such as the finite model property, tabularity, local tabularity, interpolation, amongst others.

In previous work [2], the framework of Moraschini [13], identifying translations with canonically chosen adjunctions, was used to identify a class of particularly nice translations. These were there called "sober translations", and were defined by having an associated adjunction $F \vdash G$ with the following properties:

(1) The unit of the adjunction is an isomorphism;
(2) F preserves injective homomorphism;
(3) The counit of the adjunction is an injection;
(4) G preserves surjective homomorphisms.

In particular, it was shown that any appropriate translation corresponding to such an adjunction will have a natural "Blok-Esakia theory" – a theory of "modal companions", relating extensions of one logical system with the other, and preserving and reflecting several basic properties. This constitutes a non-trivial weakening of the classical "Blok-Esakia theorem" [8, 6] for the GMT translation, which proves that the lattice of extensions of intuitionistic logic IPC is isomorphic to the lattice of normal extensions of S4.Grz modal logic.

In this paper, we apply this framework to the analysis of the Goldblatt translation. Our results emphasize that the situation is quite different for the Goldblatt translation. Using elementary observations on the lattice of extensions, we show

Theorem 2. *There can be no isomorphism between the lattice of extensions of $\mathcal{O}$ and some lattice of normal extensions of $L \supseteq$ **KTB** for L a normal extension of* **KTB**.

Furthermore, we show that in fact a reasonable Blok-Esakia theory of the sort discussed before cannot be developed. Using Goldblatt [10, 11] and Bimbo's [5] duality between orthospaces and ortholattices, we are able to show that:

Theorem 3. *The adjunction between ortholattices and* **KTB***-modal logic does not have an isomorphic unit.*

The crucial reason for this can be illustrated with the two graphs in Figure 1: even though they are different KTB-frames, with distinct logics, they correspond to the same ortholattice when seen as orthospaces.

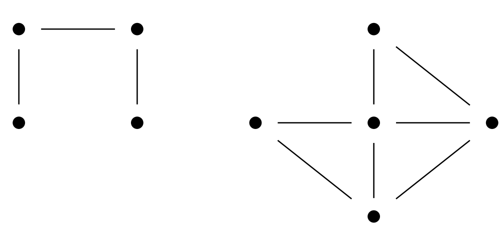

Figure 1. Two Benzene frames

However, an analysis of the algebraic and relational models at play hints at the existence of some form of translation for limited fragments. Based on the work of Gehrke and Van Gool [9], we introduce a different duality for ortholattices and specific kinds of maps.

Definition 4. *Let* $\mathbf{L}$ *be a lattice. A finite subset* $S \subseteq \mathbf{L}$ *is said to be have an admissible join if for each* $a \in L$, $a \wedge \bigvee S = \bigvee_{m \in S} a \wedge m$. *We say that a filter* F *of* L *is quasi-prime if whenever* $\bigvee S \in F$ *is an admissible join, then there is some* $m \in S$ *such that* $m \in F$.

We say that a lattice homomorphism f *is an admissible homomorphism if for each finite subset* $S \subseteq \mathbf{L}$, *then* $f[S]$ *has an admissible join whenever* S *has an admissible join.*

We then identified a full subcategory of the category orthospaces, deemed "slim orthospaces", and managed to show

Theorem 5. *There is a dual equivalence between the category of ortholattices with admissible homomorphisms and the category of slim orthospaces.*

Using these tools, our main positive result lies in the introduction of an extension of orthologic and ortholattices, adding infinitely many implication connectives to the basic signature, obtaining *orthoimplicative systems* and an associated *Orthoimplicative Logic*, which is a conservative extension of minimal orthologic. These implication connectives are of the form

$$a \hookrightarrow (b_0, ..., b_n),$$

with the meaning that it should be true at a world x if in all worlds y, consistent with x, if a holds, then one of b_i should hold. The need for infinitely many connectives arises essentially from the fact that regular subsets are not in general closed under union, and hence, it is necessary to externally model the disjunctive reasoning inherent in the implication. These operations can be defined in numerous ortholattices, including finite ortholattices, and the lattice of subspaces of a Hilbert space, and appear naturally as axiomatisations of the "Kripke implication" already discussed by Dalla Chiara [7].

We thus introduce *Orthoimplicative logic*, a calculus for these structures which is conservative over orthologic. Our axiomatisation of such structures relies on non-standard Π_2-rules as discussed and introduced in [3, 4], which relate to the requirements on admissibility. In parallel, we introduce an extension of **KTB** deemed *Sober KTB*, obtained by adding the following rule:

- If $M \subseteq \mathcal{L}_{KTB}$ is a set of formulas which is admissible, in the sense that for each ψ we can prove that:

$$\vdash_L (\bigvee_{\chi \in M} \Box\Diamond\chi) \wedge \Box\Diamond\psi \to \Box\Diamond(\bigvee_{\chi \in M} \Box\Diamond\chi \wedge \Box\Diamond\psi)$$

  then:

$$\vdash_L \Box\Diamond(\bigvee_{\chi \in M} \Box\Diamond\chi) \leftrightarrow \bigvee_{\chi \in M} \Box\Diamond\chi.$$

We then restrict our attention to axiomatic extensions of **KTB** which are closed under this rule, as well as axiomatic extensions of orthoimplicative logic closed under the non-standard rules present there. Algebraically, this corresponds to looking at *relative varieties over the fixed* Π_2*-first order formulas*: classes of algebras satisfying such a formula which are closed under homomorphic images, subalgebras and products. With trivial modifications, the results of [2] hold in this setting as well, and hence, to obtain a translation which preserves and reflects good properties and is general, it suffices to show the existence of an adjunction between the relevant categories. This is our final and key set of results:

Theorem 6. *There is an adjunction between the category of orthoimplicative systems and the category of sober KTB algebras.*

Theorem 7. *There is a surjective homomorphism from the lattice of relative varieties of KTB algebras to the lattice of relative varieties of orthoimplicative logics.*

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