Iterated structures are not modally definable

P. Balbiani and L. Uridia

[1] Université de Toulouse, Philippe.Balbiani@irit.fr
[2] TSU A. Razmadze Mathematical Institute l.uridia@freeuni.edu.ge

1 Introduction

Iterated structures are bi-relational Kripke frames where each relation is a strict partial order and additionally second relation is the greatest fixed point of the first relation under monotone operator on the lattice of all partial orders included in the first relation. The study of these structures closely relates to the modal logic study of Cantor Bendixon rank. The details about this connection as well as important results around this study could be found in [2]

In this work we touch the issue of modal definability of iterated strict partial orders. The modal logic of iterated strict partial orders is studied in [2] where we present the complete axiomatisation of the logic. The logic is axiomatised by the following modal formulas $\Box\phi \to \Box\Box\phi$, $\Box^\star\phi \to \Box^\star\Box^\star\phi$, $\Box\phi \to \Box^\star\phi$, $\Box^\star\phi \to \Box\Box^\star\phi$, $\Box^\star\phi \to \Box^\star\Box\phi$, $\Box\Box^\star\phi \to \Box^\star\phi$. Moreover we prove that the class of iterated structures is not first order definable and that the modality $\Box^*$ interpreted on the second relation is not modally definable. In this paper we prove that the class of iterated structures is not modally definable either.

2 Preliminaries

A *strict partial order* on X is a binary relation R on X such that: (i) for all $x \in X$, $x \notin R(x)$, (ii) for all $x \in X$, $R(R(x)) \subseteq R(x)$, where $R(x) = \{y \mid xRy\}$. Let $\leq$ be the binary relation between strict partial orders on X defined by $R \leq R'$ iff $R \subseteq R'$. Given a strict partial order R on X, let L_R be the set of all strict partial orders R' on X such that $R' \leq R$. We point out that the least element of L_R is the strict partial order $\emptyset$ and the greatest element of L_R is R. Moreover, the least upper bound of a family $\{R'_i\colon i \in I\}$ in L_R is the transitive closure of $\bigcup\{R'_i\colon i \in I\}$ and the greatest lower bound of a family $\{R'_i\colon i \in I\}$ in L_R is $\bigcap\{R'_i\colon i \in I\}$. Hence, $(L_R, \leq)$ is a complete lattice.

For a given a strict partial order R on X, let θ_R be the function $\theta_R\colon L_R \to L_R$ such that for all $R' \in L_R$, $\theta_R(R') = R \circ R'$, i.e. $\theta_R(R')$ is the binary relation on X such that for all $x, y \in X$, $x\theta_R(R')y$ iff there exists $z \in X$ such that xRz and $zR'y$. It is straightforward to check that the function θ_R is monotone. Since $(L_R, \leq)$ is a complete lattice, the function θ_R has a least fixpoint $\mathrm{lfp}(\theta_R)$ and a greatest fixpoint $\mathrm{gfp}(\theta_R)$. Obviously, $\mathrm{lfp}(\theta_R) = \emptyset$ while $\mathrm{gfp}(\theta_R)$ is the least upper bound of the family $\{R'\colon R' \leq \theta_R(R')\}$ in L_R [1]. Equally $\mathrm{gfp}(\theta_R)$ can be obtained by iteration of θ_R. For all ordinals α, we inductively define $\theta_R{\downarrow}\alpha$ as follows:

- $\theta_R{\downarrow}0$ is R,
- for all successor ordinals α, $\theta_R{\downarrow}\alpha$ is $\theta_R(\theta_R{\downarrow}(\alpha - 1))$,
- for all limit ordinals α, $\theta_R{\downarrow}\alpha$ is the greatest lower bound of the family $\{\theta_R{\downarrow}\beta\colon \beta \in \alpha\}$ in L_R.

The next result is, a consequence of Tarski's fixpoint theorem [1]:

Fact 1. *(i) for all ordinals α, gfp$(\theta_R) \leq \theta_R{\downarrow}\alpha$,*

(ii) there exists an ordinal α such that gfp$(\theta_R) = \theta_R{\downarrow}\alpha$.

The least ordinal α such that $\theta_R{\downarrow}\alpha = \text{gfp}(\theta_R)$ is called the *Cantor-Bendixson rank of R.*

3 A modal logic

In this section, we present a modal logic with modal operators $\Box$ and $\Box^\star$. We define the relational semantics where $\Box$ and $\Box^\star$ are respectively interpreted by strict partial orders and the greatest fixpoints of the θ-like functions they define. The language of the modal logic is defined using a countable set of propositional variables. We inductively define the set of *formulas* as follows: $\phi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi \mid \Box^\star\phi$. The other connectives are defined as usual. We obtain the formulas $\Diamond\phi$ and $\Diamond^\star\phi$ as abbreviations: $\Diamond\phi ::= \neg\Box\neg\phi$, $\Diamond^\star\phi ::= \neg\Box^\star\neg\phi$.

Relational semantics of the modal logic is based on the iterated structures which are birelational strict partial orders with certain interconnection between the two relations.

Definition 2. An *iterated structure* is a structure of the form $\mathcal{F} = (X, R, S)$ such that (i) X is a nonempty set, (ii) R is a strict partial order on X, (iii) S is the greatest fixpoint of the function θ_R in L_R.

The second relation i.e. the greatest fixpoint S of θ_R in the iterated structure $\mathcal{F} = (X, R, S)$ can be represented in more intuitive way. The following proposition gives the desired representation. This representation plays an important role in the proof of the main theorem.

Proposition 3. *For a given iterated structure $\mathcal{F} = (X, R, S)$ and two points $x, y \in X$ we have: xSy iff there exists an infinitely ascending chain $a_1, a_2, \ldots$ such that $x = a_1$ and for each i the member of chain a_i is a predecessor of y i.e. a_iRy.*

On the figure we see an iterated structure (X, R, S) where the root x satisfies the formula $\Diamond^*\phi$. The left arrow represents the relation S while small arrows represent the relation R.

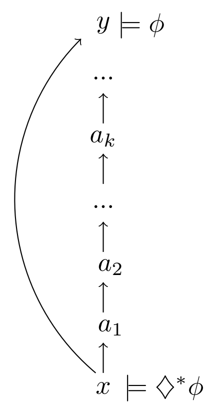

Figure 1:

4 Modal Definability

In this section we touch the main issue of our study and prove the main theorem that the class of iterated structures is not definable in modal language. In [2] we prove that the class of iterated structures is not first order definable. Besides we show that the $\Box^*$ modality is not definable in simpler language which only contains $\Box$. Here we present the more intuitive proof of this fact and additionally show that the property of S to be the greatest fixpoint of R in the bi relational frame (X, R, S) with R being strict partial order, is not definable in modal language with two modalities.

Fact 4. *[2] The modality $\Box^*$ is not definable in a simpler language which only contains $\Box$ and Boolean connectives. Which means that there is no formula ϕ in the language with one $\Box$ such that $\Box^* p \leftrightarrow \phi$.*

Theorem 5. *The class of iterated structures is not modally definable.*

Sketch. We construct the two structures $(X, R, S = \emptyset)$ and $(X', R', S' = \emptyset)$ as depicted on the picture (ignore the rounded arrows of the frame on the rigth). The first is an iterated structure and (X', R', S') is a p-morphic image of (X, R, S) while the later is not the iterated structure. This follows from the Proposition 3. p-morphism f is established by the following clauses $f(a_i) = a'_i$ and $f(b_i) = a'_\omega$.

On the figure we see an iterated structure (X, R, S) on the left and bi relational frame (X', R', S') on the right. For (X', R', S') to be an iterated structure S' should contain all pairs connected by rounded arrows

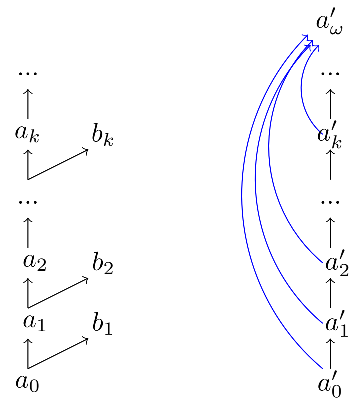

Figure 2:

$\Box$

References

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[2] P. Balbiani and L. Uridiaia, Completeness and Definability of a Modal Logic Interpreted over Iterated Strict Partial Orders. Advances in Modal Logic 2012: 71-88.