Incomplete Information and Justifications*

Dragan Doder[1], Zoran Ognjanović[2], Nenad Savić[3], and Thomas Studer[3]

[1] Université Paul Sabatier – CNRS, IRIT, Toulouse, France
dragan.doder@irit.fr
[2] Mathematical Institute of Serbian Academy of Sciences and Arts, Belgrade, Serbia
zorano@mi.sanu.ac.rs
[3] Institute of Computer Science, University of Bern, Bern, Switzerland
{nenad.savic,thomas.studer}@inf.unibe.ch

1 Introduction

Since the seminal paper about justification logics was published, [2], a whole family of justification logics has been established, including logics with uncertain justifications, see [4, 5, 7, 8].

The main feature of justification logics are formulas of the form $t : \alpha$ meaning that t *justifies* α. We are interested in logics with incomplete information distuinguishing the following three cases how $t : \alpha$ can contain incomplete information:

1) *"t" is incomplete.*
   A friend tells me that she read in *some* newspaper that α is true. I know that she reads only newspapers A and B and that newspaper B provides more reliable information than A meaning that if α was read in A, my degree of belief is equal to r and if α was read in B, my degree of belief is equal to s, where $r < s$. As a consequence of incomplete justification t (she read in *some* newspaper and did not specify in which one), my degree of belief that α is true lies in an interval $[r, s]$.
2) *":" is incomplete.*
   I see a friend across the street and shout out to him. Another person, standing close to him turns her head. The reason why she turned her head can be that she saw something in that direction or she thought that I was calling her. In this case, both t (I shout) and α (she turned her head) are clear, the only thing that is questionable is if t is the justification for α and thus my degree of belief for the whole formula $t : \alpha$ belongs to some interval.
3) *"α" is incomplete.*
   Throwing a stone over the wall towards two glass bottles and then hearing a crack sound tells me that either one of the two bottles cracked or both of them. In this case, t (I threw a stone) is certain, as well as ":" (stone hit bottle(s)). The incompleteness arises from the fact that formula α is of the form $\beta \vee \gamma$ since we do not know which bottle cracked.

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The third case can be formalized by extending the justification logic J, see e.g. [6], by a list of unary operators, $P_{\geq s}$, with an intended meaning 'the probability is greater or equal to s'. Thus, saying that justification t is a justification for a formula $\alpha \equiv \beta \vee \gamma$ can be represented by associating the probabilities r to β and s to γ. Written in this language it would have the following form: $t : P_{=r}\beta$ and $t : P_{=s}\gamma$. This formalization has already been done in [5].

In this paper we formalize the first two cases. Namely, we provide a new logic, ILUPJ[4], as an extension of the justification logic J with two families of unary operators $L_{\geq s}$ and $U_{\geq s}$, for $s \in \mathbb{Q} \cap [0, 1]$. The intended meanings of these operators are that 'the lower (upper) probability is greater or equal to s'. Therefore, saying that our degree of belief lies in an interval $[r, s]$ is represented by saying that the lower probability is equal to r and the upper probability is equal to s.

The first case, when "t" is incomplete and therefore our belief that α is true belongs to an interval $[r, s]$ we can represent in the logic ILUPJ with $t : L_{=r}\alpha$ and $t : U_{=s}\alpha$. The second case, when ":" is incomplete, i.e., situations in which we are not sure if t is the justification for α, can be represented by $L_{=r}(t : \alpha)$ and $U_{=s}(t : \alpha)$.

Semantically, lower and upper probabilities are captured as follows: For a given set of finitely additive probability measures, P, the upper probability of an event X is given by the function

$$P^*(X) = \sup\{\mu(X) \mid \mu \in P\},$$

and the lower probability of an event X is given by the function

$$P_*(X) = \inf\{\mu(X) \mid \mu \in P\}.$$

Models are Kripke-style models where we assign to each world a (lower and upper-)probabilistic space, that is a non-empty set of worlds equipped with:

a) an algebra;
b) basic J_{CS}-evaluations;
c) a *set* of finitely additive probability measures.

Using Anger and Lembcke's characterization of upper and lower probabilities with a finite number of properties, [1], we provide an axiomatization of our logic similar to the axiomatization of the logic ILUPP, see [3]. The difference lies in the fact that we need to take care that all axioms and inference rules of the logic J are included. The soundness theorem is proved in a straightforward way and for the strong completeness theorem we use a strategy that is a combination of the completeness proofs for the logics J and ILUPP, [3, 7, 9]. Namely we:

1) Prove the Deduction Theorem, as well as a few auxiliary lemmas.

[4] I stands for iterations, LUP for lower and upper probabilities and J for the justification logic J.

2) Prove Lindenbaum's Lemma: Every consistent set of formulas can be extended to a maximal consistent set.
   Due to the fact that two infinitary rules are present in our logic, we have to modify Lindenbaum's construction in the following way: If the current theory is inconsistent with the current formula derived by one of the infinitary rules, then one of the premises must be blocked.
3) Prove strong completeness by a canonical model construction.

Our last goal is to prove that the logics PJ and PPJ, from [5] and [4], respectively, are special cases of the logic ILUPJ. From the semantic point of view, it is clear that our semantics is a generalization of the semantics of PJ and PPJ since we have sets of finitely additive probability measures. Thus, setting that these sets must be singletons, we obtain the models of PJ and PPJ.

Axiomatization is a bigger challenge. Namely, the idea is to add an additional axiom of the form

$$\vdash U_{\geq r}(t : \alpha) \to L_{\geq r}(t : \alpha),$$

basically saying that these two operators coincide (since it can easily be proved that $\vdash L_{\geq r}(t : \alpha) \to U_{\geq r}(t : \alpha)$). In that sense we have an extension of the logic J with "one" operator and the idea is to infer all of their axiom from ours.

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