Non-classical probabilities in four valued framework

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One of the motivations for (some of) the substructural logics is to create a framework for modelling reasoning in more realistic contexts. A classic example is Dunn-Belnap logic, which was held by Belnap (1992) to be how a computer should think. Computers work with data and data contained in databases are usually incomplete and often contradictory. But classical logic does not seem to be an appropriate tool to work with such a kind of data. (As Michael Dunn points out in one of his articles: If the database FBI contains the fact that my car is blue and my car is red I certainly do not want them to infer that I am the enemy No 1.)

Belnap and Dunn introduced a four valued propositional logic to deal with incomplete or contradictory information. A proposition φ in their framework might be not only True or False, but also Neither (database contains no information about φ) or Both (database contains information that φ and information that $\neg\varphi$). There is an alternative approach – to maintain classical truth values T,F and to define evaluation as a set function into $\mathscr{P}(\{T,F\})$

A first attempt to generalize the Belnap-Dunn logic probabilistically has been undertaken by Dunn (2010) and Childers, Majer, Milne (20XX). Michael Dunn's four valued probability is a function which assigns to each event A a four valued vector (*belief, disbelief, uncertainty, conflict*) of non-negative numbers. The vector is normalized, i.e. its entries add up to one. No further dependencies between the four entries are assumed.

(Childers, Majer, Milne , 20XX) define probability directly over the propositions of Dunn Belnap logic. Their probability does not satisfy the classical Kolmogorovian axioms, but their weaker version (cf. also Priest (2006) or Mares (2014)):

A1 $0 \leq p(\varphi) \leq 1$ (normalization)

A2 $\varphi \vDash_L \psi$ then $p(\varphi) \leq p(\psi)$ (monotony with respect to logical consequence)

A3 $p(\varphi \wedge \psi) + p(\varphi \vee \psi) = p(\varphi) + p(\psi)$ (inclusion/exclusion principle),

where $\vDash_L$ denotes the logical consequence relation of Belnap-Dunn logic. These axioms give rise to a non-standard notion of probability. In particular probabilities of φ and $\neg\varphi$ are not complementary in the usual sense. Instead, they are connected by a much weaker condition:

$$p(\varphi \wedge \neg\varphi) + p(\varphi \vee \neg\varphi) = p(\varphi) + p(\neg\varphi),$$

which allows for $p(\varphi \wedge \neg\varphi) > 0$ (positive probability of gluts) and $1 - p(\varphi \vee \neg\varphi) > 0$ (positive probability of gaps).

Both approaches are intertranslatable. The relation of "probability" (of φ) in the latter approach to "belief" in Dunn's setting, (i.e. b in the vector (b,d,u,c)) is the same as the relation of "at least true" to "exactly true" in the standard relevant logic, where, A is "exactly true" when $T \in v(A)$ and $F \notin v(A)$. Similarly, Dunn's b is a "pure" belief in A , and can hence be expressed as

$$b(A) = p(A) - p(A \wedge \neg A)$$

where p stands for the second approach's probability function. Similar translations apply to disbelief and the remaining components:

$$d(A) = p(\neg A) - p(A \wedge \neg A) \quad c(A) = p(A \wedge \neg A) \quad u(A) = 1 - p(A \vee \neg A)$$

It is easy to check that the b, d, u and c hence defined sum up to unity. Dunn's approach, however, is more expressive than the Prague setting. In translating from the double valuation approach, at least one of the conflict or uncertainty component has value 0 after the translation. Lastly, we remark that this translation also works also in the other direction. Defining p as

$$\begin{array}{ll} p(A) = b_A + c_A & p(\neg A) = d_A + c_A \\ p(A \vee \neg A) = 1 - u_A & p(A \wedge \neg A) = c_A \end{array}$$

Besides clarifying the relation between the two approaches introduced above, this contribution expands either non-standard probability framework by considering logical relationships between different formulas. Being informed about the probabilities of φ and ψ, we may for instance ask about the (non-standard) probability of $\varphi \wedge \psi$. Relatedly, we may be interested in what happens if the agent learns ψ, i.e. we may ask about $p(\varphi|\psi)$. Lastly, we may inquire into combining probabilities of different sources. That is, if two agents differ in their non-standard probabilities of φ, we can ask about ways for combining these into a joint belief.

Notably, the frameworks discussed have remained largely silent about these questions. Dunn's 2010, for instance, suggested a definition for the non-standard probability $\varphi \wedge \psi$. This definition assumed φ and ψ to be probabilistically independent, irrespective of their exact form or content. This raises various problems. For instance, $p(\varphi \wedge \varphi)$ need not be the same as $p(\varphi)$, as the former treats both occurrences of φ as probabilistically independent from each other.

In the current contribution, we suggest an alternative, semantically based approach to the probability of conjunctions $p(\varphi \wedge \psi)$. In a generalization of Bayes' rule, we moreover show how such conjunctive belief relates to conditional belief. As it turns out, defining conditional beliefs requires to disambiguate various possible readings of the event learned. Within classic probabilistic reasoning, learning ψ and forming $p(\varphi|\psi)$ is a shorthand for learning that ψ is *true*. Learning that ψ is false, similarly, relates to $p(\varphi|\neg\psi)$. In a generalized setting, we could learn more than the truth or falsity of ψ: we could, for instance, learn that it is at least true, at least false, exactly true, true and false.... We define a notion of conditional belief for each of these possible updating events. Moreover, we also explore various approaches to settling disagreement between non-standard probabilistic beliefs, i.e. for merging two non-standard probability assignments on φ.

Lastly, we relate our discussion to the underlying logical spaces. As is well known, every classical probability assignment over a finite propositional language can be translated into a probability assignment over the set of valuations of that language's atoms. We explore generalizations of this fact to non-standard probabilities. In particular, we will show that non-standard probability distributions over a language with a finite set of atoms At naturally correspond to a classic probability assignment over the set of non-standard valuations, that is over

$$\{q \mid q \in \mathsf{At}\} \cup \{\neg q \mid q \in \mathsf{At}\}$$

More specifically, if we define the truth sets $[\cdot]$ of formulas φ as

$$\begin{array}{ll} [p] = \{x \mid p \in x\} & [\neg p] = \{x \mid \neg p \in x\} \\ [\varphi \wedge \psi] = [\varphi] \cap [\psi] & [\varphi \vee \psi] = [\varphi] \cup [\psi] \\ [\neg(\varphi \wedge \psi)] = [\neg\varphi \vee \neg\psi] & [\neg(\varphi \vee \psi)] = [\neg\varphi \wedge \neg\psi] \end{array}$$

for p atomic, we get that each classic probability assignment μ on the set of non-standard valuations induces a probability function p_μ on $\mathscr{L}(\mathsf{At})$, the logical language over At generated by the propositional connectives, via

$$p_\mu(\varphi) := p([\varphi]).$$

This induced probability function p_μ satisfies axioms [A1]-[A3] and is hence a non-standard probability function in the sense defined above.

In fact, the converse also holds true: Every non-standard probability function p in the sense defined above can be represented as a μ_p for some μ on the set of non-standard valuations for the corresponding language. Hence, [A1]-[A3] is a sound and complete axiomatization with respect to the class

$$\left\{ (M_0, \mu) \mid \mu : M \to [0,1], \sum_{m \in M} \mu(m) = 1 \right\}$$

where M_0 is the set of non-standard valuations over the set of atoms At, i.e. $M_0 := \mathscr{P}(\{q \mid q \in \mathsf{At}\} \cup \{\{\neg q \mid q \in \mathsf{At}\}\})$.

If time permits, we will moreover outline a theory of aggregation over non-standard probabilities. That is, we consider settings where two expert advisers provide different non-standard probabilities for some φ. The question of aggregation, then, is which probabilities of φ to adopt in light of these differing inputs. Within classic probability theory, options are fairly limited. One could either decide to follow one of the agents, or build a weighted average between the two, possibly discarding one of the sources altogether. In non-standard probability, in contrast, further options arise. One could, for instance take a credulous approach, setting the new probabilities of φ and $\neg\varphi$ to be the maximum of the input probabilities of φ or $\neg\varphi$ respectively. Likewise, a defensive approach would suggest to set the new probabilities to the minimum of the input probabilities of φ or $\neg\varphi$ respectively. Time permitting, we will characterize various such options and relate them to various logical operations in the Belnap-Dunn framework.

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