On negation based on a class of absurdities

Lex Hendriks and Dick de Jongh
ILLC, Universiteit van Amsterdam

1 Introduction

In [4] Haskell Curry suggested an approach to negation based on a class of 'counteraxioms' or 'refutable' propositions. Instead of the usual definition of negation as $\neg A = A \to \bot$, one imagines a series of counteraxioms f_i, and then define that $\Gamma \vdash \neg A$ iff $\Gamma \vdash A \to f_i$ for some f_i. Curry did not really work this out in his book but continued with the well-known case of one such refutable proposition f, which yields Johansson's Minimal logic, MPC.

In the early nineties Lloyd Humberstone conjectured that adding a negation of the form $\neg A = \bigvee(A \to f_i)$ to the positive (minimal or intuitionistic) proposition logic, PPC, would be equivalent to adding the axiom of contraposition, $(p \to q) \to (\neg q \to \neg p)$, to PPC. This conjecture was proved by Allen Hazen in [5]. Hazen called the logic *Subminimal Logic.*

Unaware of Hazen's thesis or Humberstone's conjecture, in [3] Colacito, de Jongh and Vargas introduced a neighborhood semantics for this logic, which they called CoPC, *contraposition logic.* Hazen's proof of his theorem is rather sketchy. In this paper we will a give a proof based on the completeness of CoPC w.r.t. finite neighborhood frames making use of Hazen's ideas.

2 Contraposition Logic

Following [3], for the semantics of CoPC we use *neighborhood frames*, i.e. Kripke frames (partially ordered sets) with a function assigning an upset to each upset in the frame (see [1] for a comparison of this formulation of neighborhood semantics with the usual one).

Definition 1 *Let $\mathcal{U}(W)$ be the set of upsets of $\langle W, \leq\rangle$. A function $\mathcal{N}$: $\mathcal{U}(W) \longrightarrow \mathcal{U}(W)$ is called*

local	*if*	*for all* $X, Y \in \mathcal{U}(W)$ $\mathcal{N}(X) \cap Y = \mathcal{N}(X \cap Y) \cap Y$,
antitone	*if*	$X, Y \in \mathcal{U}(W)$ *and* $X \subseteq Y$ *implies* $\mathcal{N}(Y) \subseteq \mathcal{N}(X)$.

Definition 2 *A structure* $F = \langle W, \leq, \mathcal{N} \rangle$ *is a* CoPC-model if:

$\langle W, \leq \rangle$	is a partially ordered set,
$\mathcal{N}$	a function, $\mathcal{N} : \mathcal{U}(W) \longrightarrow \mathcal{U}(W)$, both *local* and *antitone*,
V	a function, V: PROP $\longrightarrow \mathcal{U}(W)$, PROP a set of atoms.

F is called *tree-like* if for every $w \in W$ the set $\downarrow w = \{v \in W \mid v \leq w\}$ is linearly ordered.

In a CoPC-model we define $[\![A]\!] = \{w \in W \mid w \models A\}$ as usual in Kripke models for atoms and $\wedge, \vee, \rightarrow$, and stipulate: $[\![\neg A]\!] = \mathcal{N}([\![A]\!])$.

One easily extends the notion of a *p-morphism* φ, between Kripke models M and M' to CoPC-models (as in [2]) by defining:

$$w \in \mathcal{N}(X) \Leftrightarrow \varphi(w) \in \mathcal{N}'(\varphi(X)).$$

We will make use of p-morphisms to prove several operations on CoPC-models to be 'innocent', i.e. resulting in models where corresponding nodes force the same CoPC-formulas, for example to prove the following theorem, slightly extending [2], Theorem 3.1.6.

Theorem 3 *The logic* CoPC *is sound and complete for (finite) tree-like* CoPC*-models.*

3 $\mathbb{Q}$-models

In [6] Humberstone introduces $\mathbb{Q}$-frames as a generalisation of the Q-frames for MPC introduced by Segerberg in [7]. Segerberg's idea for Kripke models suitable for minimal logic is quite simple. Instead of taking $[\![\bot]\!] = \emptyset$, as in intuitionistic logic, let f be the falsum of MPC and let $[\![f]\!]$ be an arbitrary upset of the frame. $\mathbb{Q}$-frames are a generalization of this idea for a collection $\mathbb{Q}$ of such falsa. The definition of $w \models A$ in $\mathbb{Q}$-models is as usual, but for:

$$w \models \neg A \quad \Leftrightarrow \quad \exists X \in \mathbb{Q}\ \forall v \geq w\ (v \models A \Rightarrow v \in X)$$

In case $\mathbb{Q}$ is a singleton the resulting $\mathbb{Q}$-model will be an MPC model. As Humberstone observes in [6], Hazen's theorem is equivalent to the statement that CoPC is sound and complete for $\mathbb{Q}$-models.

For the proof of the completeness theorem we will use *hybrid* models, i.e. models which are both CoPC-models and $\mathbb{Q}$-models. To distinguish the two negations we will use $\sim A$ for the negation of A in the $\mathbb{Q}$-model and $\neg A$ in the CoPC-model.

Lemma 4 *For every finite* CoPC*-model M, for the language $\mathcal{L}_C$, there is a hybrid model M' such that:*

- *M is a p-morphic image of M',*
- *in $M' \models \sim A \leftrightarrow \neg A$ for all $A \in \mathcal{L}_C$.*

This immediately gives Hazen's theorem.

Theorem 5 CoPC *is sound and complete for $\mathbb{Q}$-models.*

References

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