Lattices of Intermediate Theories via Ruitenburg's Theorem*

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1. Introduction

We propose a systematic study of a class of lattices of intermediate theories stemming from Ruitenburg's Theorem, among which we can find the lattice of negative variants of intermediate logics.

Negative variants ([3, 7]) are theories obtained from intermediate logics by adding the clause $\neg\neg p \leftrightarrow p$ and closing under modus ponens, thus requiring that the $\vee$-free fragment of the logic behaves classically. *Inquisitive logic* `InqB` ([5]), characterizable as the negative variant of Medvedev's logic, is an example of such logics extensively studied in the literature ([3, 4, 7]).

Building on Ruitenburg's Theorem and a recent algebraic approach to `InqB` ([1]), we define the lattices of χ-*variants*, generalizing negative variants, and develop a semantic framework to study them. In particular, we will develop an algebraic semantics for these lattices, based on the algebraic semantics introduced in [1]. For each lattice we will also define a suitable concept of variety, obtaining a correspondence result in the style of Birkhoff.

As an application of this framework, we will show that negative logics extending `InqB` are linearly ordered under inclusion. To prove this, we will define and study a suitable generalization of locally-finite varieties in the framework of χ-variants.

2. χ-variants

Given an intermediate logic L, its *negative variant* is defined as the set $L^{\neg} := \{\phi(\overline{p}) \mid \phi[\neg\overline{p}/\overline{p}] \in L\}$[1]. As shown in [3]§3.2, negative variants can be axiomatized by a special class of Hilbert-style derivation systems, since $L^{\neg} = \mathrm{MP}(L \cup \{\neg\neg p \leftrightarrow p \mid p \text{ atomic proposition}\})$[2]—compare with Definition 3.2.13 of [3]. This identity justifies the intuition that atoms behave like $\neg\neg$-fixpoints for negative variants.

A surprising result by Ruitenburg allows us to generalize the concept of negative logic, following the fixpoint intuition. Let $\chi(p)$ be a formula in one propositional variable. Ruitenburg's Theorem ([8]; see also [6, 9] for a recent semantic proof) states that the sequence $p, \chi(p), \chi^2(p) := \chi(\chi(p)), \dots$ is definitely periodic with period 2, modulo logical equivalence; that is, there exists an n such that $\vdash \chi^n(p) \leftrightarrow \chi^{n+2}(p)$, where $\vdash$ is intuitionistic provability. We will call the smallest n for which this holds the *index of* χ.

We define the χ-*variant* of an intermediate logic L as $L^{\chi} := \{\phi(\overline{p}) \mid \phi[\chi^n(\overline{p})/\overline{p}] \in L\}$, for n the index of χ. Intuitively, we are requiring atomic proposition to play the role of χ^2–fixpoints, in analogy with the case of negative variants. And in fact, we obtain an analogous axiomatization result: $L^{\chi} = \mathrm{MP}(L \cup \{\chi^2(p) \leftrightarrow p \mid p \text{ atomic proposition}\})$.

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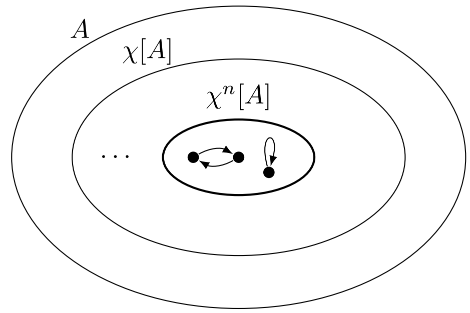
[1] With $\overline{p}$ we denote a sequence of variables $p_1, \dots, p_k$. Given $\chi(q)$ a formula with only one free variable, we denote by $\chi(\overline{p})$ the sequence $\chi(p_1), \dots, \chi(p_k)$. We denote by $\phi[\overline{\psi}/\overline{p}]$ the formula obtained by substituting every instance of p_i with the formula ψ_i.

[2] MP denotes the closure under modus ponens.

3. Algebraic semantics for χ-variants

An algebraic semantics for `InqB` was presented in [1]. Following the same approach, we can define a semantics for χ-variants. The key to this generalization lies in an algebraic interpretation of Ruitenburg's Theorem.

Given an Heyting algebra A, we can treat χ as an endomorphism of A, mapping $a \in A$ to $[\![\chi(p)]\!]_{[p \mapsto a]}$. Ruitenburg's Theorem tells us exactly that χ restricted to the set $A^\chi = \chi^n[A]$ is an *involution*, and so that A^χ are exactly the χ^2–fixpoints of the algebra. Notice that, in the case of negation, $A^{\neg p}$ is exactly the set of *regular elements* of the algebra.



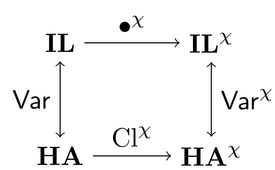
We call a valuation[3] $\sigma : \mathrm{AP} \to A$ a χ-*valuation* if the range of σ is contained in A^χ. We can define a semantics suitable to study χ-variants simply by restricting the valuations we consider to be χ-valuations: we say that ϕ is χ-valid on A if $[\![\phi]\!]^A_\sigma = \top$ for every χ-valuation σ. We will denote by $\mathrm{Th}^\chi(A)$ the set of valid formulas in A; for $\mathcal{C}$ a class of Heyting algebras, we will denote by $\mathrm{Th}^\chi(A) = \bigcap\{\mathrm{Th}^\chi(A) \mid A \in \mathcal{C}\}$.

A routine check shows that $\mathrm{Th}^\chi(\mathcal{C}) = (\mathrm{Log}(\mathcal{C}))^\chi$. In particular, every class of algebras defines a χ-variant via the semantics introduced; and every negative logic L^χ is the class of some class of Heyting algebras, since $\mathrm{Th}(\mathsf{Var}(L)) = L$[4]. Thus for every χ-variant L^χ, the semantics introduced is *correct and complete* if restricted to the appropriate class of Heyting algebras.

This result can be further refined, obtaining a correspondence between χ-variants and some special varieties, in the same style as Birkhoff Theorem (Theorem 11.9 in [2]). The key observation is that, under a χ-valuation, the semantics of a formula in A is restricted to the sub-algebra generated by A^χ. In case this sub-algebra coincides with the whole algebra, we say that A is core-generated.

If we denote by V_{CG} the set of core-generated elements of the variety V, the observation above entails that V_{CG} determines univocally $\mathrm{Th}^\chi(V)$. Moreover, if $\mathsf{V}_{CG} = \mathsf{V}'_{CG}$ then $\mathrm{Th}^\chi(V) = \mathrm{Th}^\chi(V')$. This suggests the definition of the following closure operator on varieties: $\mathrm{Cl}^\chi(\mathsf{V}) := \{B \mid \exists A \in \mathsf{V}.\, A \preceq B \text{ and } A^\chi = B^\chi\}$[5]—notice that $\mathsf{V}_{CG} = \mathsf{V}'_{CG}$ iff $\mathrm{Cl}^\chi(\mathsf{V}) = \mathrm{Cl}^\chi(\mathsf{V}')$. Indeed, it is easy to verify that given a χ-variant L^χ, the set $\mathsf{Var}^\chi(L^\chi) := \{A \mid L^\chi \subseteq \mathrm{Th}^\chi(A)\}$ is a variety closed under the operation Cl^χ.

Theorem 1. *Let* $\mathbf{IL}^\chi$ *and* $\mathbf{HA}^\chi$ *denote the lattice of* χ*-variants and the lattice of varieties closed under* Cl^χ *respectively. The function* $\mathsf{Var}^\chi : \mathbf{IL}^\chi \to \mathbf{HA}^\chi$ *is an isomorphism and* $\mathrm{Th}^\chi = (\mathsf{Var}^\chi)^{-1}$.



[3] AP denotes the set of atomic propositions.
[4] With $\mathsf{Var}(L)$ we denote the variety of Heyting algebras validating all formulas in L.
[5] With $A \preceq B$ we denote that A is a subalgebra of B.

4. Application: extensions of InqB

As an application of the framework introduced, we prove that negative logics extending InqB are linearly ordered by inclusion. By Theorem 1, this implies that subvarieties of $\mathsf{Var}^{\neg p}(\mathtt{InqB})$ are linearly ordered. An important feature, that we can exploit to prove this, is that core-generated elements of $\mathsf{Var}^{\neg p}(\mathtt{InqB})$ are locally finite—compare with Lemmas 4.1 and 4.3 of [1]. This property corresponds to the usual locally-finiteness for varieties of Heyting algebras, and can be studied more in general for varieties in $\mathbf{HA}^{\chi}$.

We say a variety $\mathsf{V} \in \mathbf{HA}^{\chi}$ is χ-locally-finite if all the algebras in V_{CG} are locally-finite; or equivalently, given $A \in \mathsf{V}$ and finitely many elements $a_1, \dots, a_k \in A^{\chi}$, the sub-algebra $\langle a_1, \dots, a_k \rangle_A$ is finite. Notice that, under a χ-valuation, the interpretation of a formula ϕ lies in the subalgebra generated by the core, so this seems a sensible generalization of locally-finiteness to study χ-logics and the corresponding varieties. And in fact we have the following:

Lemma 2. *If* V *is* χ*-locally-finite, then* V *is generated by its finite, subdirectly-irreducible, core-generated elements.*

So, since $\mathsf{Var}^{\neg p}(\mathtt{InqB})$ is $\neg p$-locally-finite, to determine its subvarieties it suffices to consider its finite, subdirectly-irreducible, core-generated elements. These are exactly (up to isomorphism) the finite *inquisitive algebras* $H(B)$ studied in [1], for B a finite Boolean algebra. In particular, denoting with B_k the unique (up to isomorphism) Boolean algebra with k generators, we have $H(B_k) \in \mathsf{Var}(\{H(B_l)\})$ for $k \leq l$.

Combining these observations with Theorem 1, we obtain that the negative variants extending properly InqB are all of the form $\mathtt{InqB}_N := \mathrm{Th}^{\neg p}(\{H(B_k) \mid k \leq N\})$ for $N \in \mathbb{N}$. Moreover, these are all distinct theories since $C_N \in \mathtt{InqB}_N \setminus \mathtt{InqB}_{N+1}$ for the following:

$$C_N \quad := \bigvee_{\substack{i \neq j \\ 1 \leq i,j \leq N+1}} (p_i \to p_j)$$

Therefore, we obtain the following characterization:

Theorem 3. *The sublattice of negative variants extending* InqB *forms the chain:*

$$\mathtt{InqB} \subsetneq \dots \subsetneq \mathtt{InqB}_3 \subsetneq \mathtt{InqB}_2 \subsetneq \mathtt{InqB}_1 = \mathbf{CPC}.$$

References

[1] N. Bezhanishvili, G. Grilletti, and W. H. Holliday. Algebraic and topological semantics for inquisitive logic via choice-free duality. In *Proceedings of WoLLIC 2019 (to appear)*, 2019.

[2] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra*. Springer, 1981.

[3] I. Ciardelli. Inquisitive semantics and intermediate logics. MSc Thesis, University of Amsterdam, 2009.

[4] I. Ciardelli. *Questions in logic*. PhD thesis, Institute for Logic, Language and Computation, University of Amsterdam, 2016.

[5] I. Ciardelli, J. Groenendijk, and F. Roelofsen. *Inquisitive Semantics*. Oxford University Press, 2018.

[6] S. Ghilardi and L. Santocanale. Free Heyting Algebra Endomorphisms: Ruitenburg's Theorem and Beyond. *arXiv e-prints*, page arXiv:1901.01252, Jan 2019.

[7] R. Iemhoff and F. Yang. Structural completeness in propositional logics of dependence. *Archive for Mathematical Logic*, 55(7):955–975, 2016.

[8] W. Ruitenburg. On the period of sequences $(A_n(p))$ in intuitionistic propositional calculus. *Journal of Symbolic Logic*, 49(3):892–899, 1984.

[9] L. Santocanale and S. Ghilardi. Ruitenburg's Theorem via Duality and Bounded Bisimulations. *arXiv e-prints*, page arXiv:1804.06130, Apr 2018.