Flat Polygonal Logics in d-Semantics

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1 Introduction

This paper is a natural follow up of a series of papers on polyhedral semantics for modal and intermediate logics. This research area became actively investigated in recent years by collaborating groups centered in Amsterdam, Milan and Tbilisi [2, 4, 5]. The main distinction of polyhedral semantics from standard topological semantics is in restricting valuation functions to range over *polyhedral subsets* of the relevant space endowed with some kind of linearity structure – polyhedra in Euclidean spaces being the prime examples.

Let $\mathfrak{B}^n$ be the Boolean subalgebra of the full powerset Boolean algebra $\wp(\mathbb{R}^n)$ of all subsets of $\mathbb{R}^n$ generated by (either open or closed) halfspaces. Elements of $\mathfrak{B}^n$ are called *polyhedral sets*. $\mathfrak{B}^n$ turns out to be closed under the topological closure or derived set operators. It is well known that these operators serve as a basis for two distinct topological interpretations for modal language. More widely known *C-semantics* treats modality (the diamond) as the closure operator of a topological space. In algebraic terms this amounts to dealing with the classes of *closure algebras*. Lesser known *d-semantics* interprets the modal diamond as the *derivative operator* of a topological space. Algebraically this amounts to the investigation of the classes of *derivative algebras*. It is straightforward that $\mathfrak{B}^n$ can be treated as a closure algebra, since the closure $\mathbf{C}(P)$ of a polyhedron P is again a polyhedron – $\mathbf{C}(P) \in \mathfrak{B}^n$. In a similar way, the set $\mathbf{d}(P)$ of all limit points of a polyhedron P is a polyhedron. To make a clear distinction between the resulting modal algebras, by $\mathfrak{B}^n$ we denote the closure algebra, while by $\mathfrak{B}^n_d$ we denote the derivative algebra of all subpolyhedra of $\mathbb{R}^n$. The C-logic $\mathsf{Log}(\mathfrak{B}^2)$ of two-dimensional polyhedra is studied and axiomatized in [4] while the d-logic of two-dimensional polyhedra $\mathsf{Log}(\mathfrak{B}^2_d)$ is studied and axiomatized in [5]. In the present contribution we are interested in polyhedral d-logics.

Recall that the modal logic $\mathbf{K4} = \mathbf{K} + \Box p \to \Box\Box p$ is the logic of transitive Kripke frames. The logic $\mathbf{K4.Grz}$ is defined as $\mathbf{K4.Grz} = \mathbf{K4} + \Box(\Box(p \to \Box p) \to p) \to \Box p$. It turns out that $\mathfrak{B}^n_d$ is a locally finite $\mathbf{K4.Grz}$-algebra.

For a relativization of $\mathfrak{B}^n_d$ to a polyhedral set $P \in \mathfrak{B}^n_d$ we will use notation P^+. We consider *polyhedral d-logics* – logics $\mathsf{Log}\{P_i^+ \mid i \in I\}$, generated by some family $(P_i)_{i\in I}$ of polyhedra $P_i \in \mathfrak{B}^{n_i}_d$. Since each $\mathfrak{B}^n_d$ is a locally finite $\mathbf{K4.Grz}$-algebra, polyhedral d-logics are extensions of $\mathbf{K4.Grz}$ and each one of them has the finite model property.

In the current work we axiomatize the largest polyhedral d-logic, i.e. the d-logic of all polyhedra. We also study in details the d-logics of polyhedra of dimension 2 or less. In particular we fully characterize *flat polygonal* d-logics, that is 2-dimensional d-logics generated by any class of polygons $P_i \in \mathfrak{B}^2_d$ embeddable inside the 2-dimensional plane $\mathbb{R}^2$.

2 Polyhedral d-Logics

Polyhedral d-logics are generated by algebras of type P_i^+ where $P_i \in \mathfrak{B}^{n_i}_d$ is a polyhedron. Each P^+ is of finite *height* and hence, locally finite [3]. This has to do with the geometric dimension of P being finite. It follows that polyhedral d-logics enjoy the finite model property and their study can be reduced to the study of the corresponding finite Kripke frames. Since P^+ is always a **K4.Grz**-algebra, its finite Kripke frames are finite *weak partial orders* i.e. frames (W, R) such that the reflexive closure R° of R is a partial order. We call such frames *w-posets*. Note that w-posets are transitive and antisymmetric.

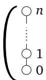
Each polyhedral d-logic L has well-defined *dimension* $\dim L$: it is either the smallest d for which L forbids the $(d+1)$-element reflexive chain, or infinity, if such a d does not exist. This happens to coincide with the maximum of the geometric dimensions of the polyhedra P which validate L. The polyhedral d-logics of finite dimension are of finite height and hence, locally finite. In the next theorem we give the axiomatization of the logic of all polyhedra in d-semantics.

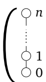
Theorem 1. *The d-logic of all polyhedra is* $\mathbf{K4.Grz} + \Box(\Box p \to p) = \mathbf{K4.Grz} + \sigma\Big($  $\Big) + \sigma\Big($  $\Big)$

Here and in what follows by $\sigma(\mathfrak{F})$ we denote *the subframe axiom* of the w-poset $\mathfrak{F}$ [6]. The depiction of w-posets follows the convention of denoting reflexive points by white circles and the irreflexive points by filled black circles.

In the following theorem we focus on the fixed dimension n and characterise/axiomatize the minimal and maximal d-logics of dim n polyhedra.

Theorem 2. *Let L be a polyhedral d-logic of dim n. Then* $\mathbf{K4.Grz}_n \subseteq L \subseteq \mathbf{PL}^d_n$*, where:*

1. *Maximal polyhedral d-logic of dim n is* $\mathbf{PL}^d_n = \mathsf{Log}(\mathfrak{B}^n_d)$
2. *Minimal polyhedral d-logic of dim n is* $\mathbf{K4.Grz}_n := \mathbf{K4.Grz} + \Box(\Box p \to p) + \sigma\Bigg($  $\Bigg)$

Where $\sigma\Bigg($  $\Bigg)$ *is the subframe axiom forbidding the $(n+1)$-element reflexive chain.*

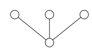
There is a single polyhedral d-logic of dim 0 – the logic of one irreflexive point characterised by axiom $\Box\bot$. Let us denote by $\mathfrak{F}^\bullet_n$ the rooted w-poset of height 2 with irreflexive root and n-many maximal reflexive points.

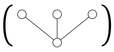
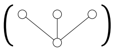
Theorem 3. *Polyhedral d-logics of dim 1 form a countable chain (under inclusion) between* $\mathbf{K4.Grz}_1$ *and* $\mathbf{PL}^d_1$ *which is presented as follows:*

$$\mathbf{K4.Grz_1} \subseteq \cdots \subseteq \mathsf{Log}(\mathfrak{F}^\bullet_n) \subseteq \cdots \subseteq \mathsf{Log}(\mathfrak{F}^\bullet_2).$$

We now turn to *flat polyhedra* – those dim n polyhedra that are embedded into the ambient Euclidean space $\mathbb{R}^n$ of the *same dimension*. The relevant algebraic notion is that of *relativization*, while the relevant modal notion is that of *downward subframization* [6], [1]. Call the polyhedral d-logic L of dim n *flat* iff L is complete wrt some class $(P_i^+)_{i\in I}$ of polyhedral derivative algebras such that $P_i \in \mathfrak{B}^n_d$ are polyhedra of dim n inside $\mathbb{R}^n$ for all $i \in I$.

Theorem 4. *The least flat polyhedral d-logic* $\mathbf{Flat}^d_n$ *of dim n is the downward subframization of* $\mathbf{PL}^d_n$.

Our main results concern the flat d-logics of dim 2 – we call them *Flat Polygonal d-Logics*. By definition, such logics are generated by a family of relativizations of $\mathfrak{B}^2_d$. In other words, flat polygonal logics are complete wrt some class $(P_i^+)_{i\in I}$ where each P_i is a flat polygon – a polygonal subset of the Euclidean plane $\mathbb{R}^2$. We will give a full characterization of flat polygonal d-logics, using an explicit collection of Jankov-Fine axioms for certain finite w-posets. It turns out that $\mathbf{Flat}^d_2$ is the logic of finite $\mathbf{K4.Grz}_2$-frames which are not up-reducible to the poset .

Theorem 5. $\mathbf{Flat}_2^d = \mathbf{K4.Grz}_2 + \chi$  *where* χ  *is the Jankov-Fine axiom forbidding the reflexive 3-fork (as an up-reduction). Flat polygonal logics are all in the interval* $[\mathbf{Flat}_2^d, \mathbf{PL}_2^d]$.

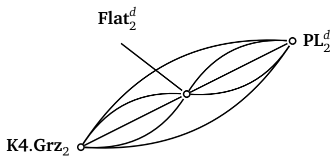


Figure 1: The (flat) polygonal logics inside the lattice of all extensions of $\mathbf{K4.Grz}_2$

To describe the flat polygonal logics occurring between $\mathbf{Flat}_2^d$ and $\mathbf{PL}_2^d$, we introduce w-posets $\mathfrak{F}^\bullet_{m,n}$ with irreflexive root depicted below that are ordered by *reducibility* – $\mathfrak{F}$ is reducible to $\mathfrak{G}$ if there exists an onto p-morphism from $\mathfrak{F}$ to $\mathfrak{G}$. The poset of these frames is depicted on Figure 2.

The reducibility among $\mathfrak{F}^\bullet_{m,n}$ can be described as follows: $\mathfrak{F}^\bullet_{m,n}$ reduces to $\mathfrak{F}^\bullet_{m',n'}$ iff $m+n \geqslant m'+n'$ and $m \geqslant m'$. Denote the poset of these frames by Q.

Lemma 6. *The dual poset of* Q *is a well partial order, i.e.* Q *contains neither infinite strictly ascending chains, nor infinite antichains.*

For every antichain α in Q the corresponding d-logic L_α is obtained by adding to $\mathbf{Flat}_2^d$ the Jankov-Fine axioms $\chi(\mathfrak{F}^\bullet_{m,n})$ for each $\mathfrak{F}^\bullet_{m,n} \in \alpha$. It is not difficult to see, that $L_\alpha \subseteq L_\beta$ iff $\alpha \subseteq \downarrow\beta$. Moreover:

Theorem 7. *The d-logics L_α, for $\alpha \subset$ Q an antichain, are all different, and exhaust all flat polygonal d-logics, that is all polygonal d-logics between* $\mathbf{Flat}_2^d$ *and* $\mathbf{PL}_2^d$.

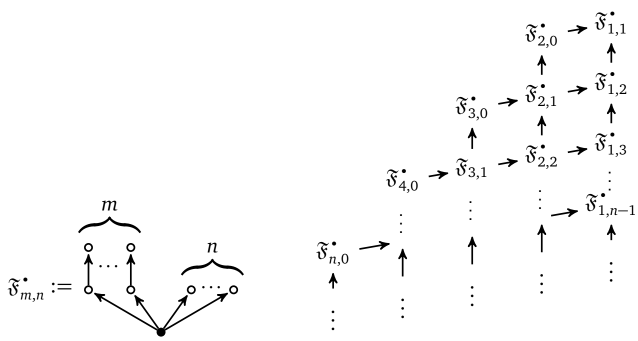


Figure 2: Poset Q of the frames $\mathfrak{F}^\bullet_{m,n}$ ordered by reducibility

It follows that there are only countably many flat polygonal d-logics, each of which is finitely axiomatizable and decidable. In the talk we will also present a way to describe the Kripke frames for each flat polygonal d-logic L based on the upset of L-frames inside Q and a certain operation on w-posets defining $\mathbf{PL}_2^d$ and $\mathbf{PL}_1^d$ – ir-crown frames [5] and n-forks with irreflexive root $\mathfrak{F}^\bullet_n$.

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