Towards a Proof Theory for Henkin Quantifiers

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Henkin introduced the general idea of dependent quantifiers extending classical first-order logic [2], cf. [3] for an overview. This leads to the notion of a partially ordered quantifier with m universal quantifiers and n existential quantifiers, where F is a function that determines for each existential quantifier on which universal quantifiers it depends (m and n may be any finite number). The simplest Henkin quantifier that is not definable in ordinary first-order logic is the quantifier Q_H binding four variables in a formula. A formula A using Q_H can be written as $A_H = \begin{pmatrix} \forall x \; \exists u \\ \forall y \; \exists v \end{pmatrix} A(x, y, u, v)$. This is to be read "For every x there is a u and for every y there is a v (depending only on y)" such that $A(x, y, u, v)$. If the semantical meaning of this formula is given in second-order notation, the above formula is semantically equivalent to the second-order formula $\exists f \exists g \forall x \forall y A(x, y, f(x), g(y))$, where f and g are function variables.

Systems of partially ordered quantification are intermediate in strength between first-order logic and second-order logic. Similar to second-order logic, first-order logic extended by Q_H is incomplete [5]. In proof theory incomplete logics are represented by partial proof systems, c.f. the wealth of approaches dealing with partial proof systems for second-order logic. In an analytic setting, these partial systems allow the extraction of implicit information in proofs, i.e. proof mining. However, in contrast to second-order logic only few results are dealing with the proof theoretic aspect of the use of branching quantifiers in partial systems.[1]

This lecture provides a globally sound but possibly locally unsound sequent calculus $\mathbf{LH}^{++}$ for Q_H, which is cut-free for a natural partial semantics. $\mathbf{LH}^{++}$ is based on the calculus $\mathbf{LK}^{++}$ of [1]. It is shown that it is impossible to construct an analytic locally sound sequent calculus for Q_H. This methodology can be extended to all Henkin quantifiers.

References

1. Juan P. Aguilera and Matthias Baaz. Unsound inferences make proofs shorter. *J. Symb. Log.*, 84(1):102–122, 2019.
2. L. Henkin. Some remarks on infinitely long formulas. In *Journal of Symbolic Logic*, volume 30, pages 167–183. Pergamon Press, 1961.
3. Michał Krynicki and Marcin Mostowski. Henkin quantifiers. In *Quantifiers: logics, models and computation*, pages 193–262. Springer, 1995.

[1] The most relevant paper is the work of Lopez-Escobar [4], describing a natural deduction system for Q_H. The setting is of course intuitionistic logic. The formulation of the introduction rule for Q_H is corresponds to the introduction rule right in the sequent calculus developed in this paper. The elimination rule lacks.

4. E.G.K. Lopez-Escobar. Formalizing a non-linear Henkin quantifier. *Fundamenta Mathematicae*, 138(2):93–101, 1991.
5. Marcin Mostowski and Konrad Zdanowski. Degrees of logics with Henkin quantifiers in poor vocabularies. *Archive for Mathematical Logic*, 43(5):691–702, 2004.