Polyhedral semantics for intermediate logics

Sam Adam-Day[1], Nick Bezhanishvili[1], David Gabelaia[2], and Vincenzo Marra[3]

[1] Institute for Logic, Language and Computation, Universiteit van Amsterdam
[2] Andrea Razmadze Mathematical Institute, Ivane Javakhishvili Tbilisi State University
[3] Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano

The collection of open subpolyhedra of any compact polyhedron forms a Heyting algebra, which leads to the polyhedral semantics of intuitionistic propositional calculus IPC [5]. A similar approach to modal logics was developed in [7]. Precursors of this work are [1], [3] and [2].

In this abstract we investigate polyhedral completeness of intermediate and modal logics. For the lack of space we will only concentrate on intermediate logics. However, all the results can be generalized to modal logics above $\mathsf{S4.Grz}$. We will define when a logic is polyhedrally complete and give a criterion for polyhedral completeness of intermediate logics in terms of the class of its finite Kripke frames. We will then use this criterion to show that many well-known intermediate logics (e.g., all stable logics) are polyhedrally incomplete. We will also use this criterion to give examples of logics that are polyhedrally complete. A full classification of polyhedrally complete intermediate and modal logics remains an open problem.

Let P be an n-dimensional compact polyhedron. By an *open subpolyhedron* of P we mean a subset of P whose complementary subset in P is a compact polyhedron. Under inclusion order, the poset $\mathsf{Sub}(P)$ of all open subpolyhedra of P is a Heyting algebra [5]. For a propositional formula φ, we say that $P \models \varphi$ if $\mathsf{Sub}(P) \models \varphi$ (i.e., φ is valid in the Heyting algebra $\mathsf{Sub}(P)$). For a class $\mathcal{P}$ of polyhedra we write $\mathcal{P} \models \varphi$ if $P \models \varphi$ for each $P \in \mathcal{P}$.

Definition 1. An intermediate logic L is *polyhedrally complete* if there is a class $\mathcal{P}$ of polyhedra such that for each formula φ we have $L \vdash \varphi$ iff $\mathcal{P} \models \varphi$.

It was shown in [5] that IPC and BD_n (the intermediate logic of frames of depth n) are polyhedrally complete. It was also proved there that $\mathsf{Sub}(P)$ is a locally finite Heyting algebra, which implies that if L is polyhedrally complete then it has the finite model property. Therefore, the logics that do not have the f.m.p. are not polyhedrally complete. We will now formulate the criterion of polyhedral completeness. As a key tool, we use a classical notion first introduced by Alexandrov in the first half of the last century in connection with his studies of posets as spaces.

Definition 2. The *nerve*, $\mathcal{N}(P)$, of a poset P is the set of all subsets of P which are linearly ordered (i.e. the set of all chains in P). We order $\mathcal{N}(P)$ by subset inclusion $\subseteq$. When P is rooted, define the *rooted nerve*, $\mathcal{N}^*(P)$, of P to be the set of all chains in P containing the root element, again ordered by subset inclusion.

Theorem 3. *An intermediate logic L is polyhedrally complete if and only if there is a class of finite rooted frames $\mathcal{C}$ closed under $\mathcal{N}^*$ such that* $\mathrm{Logic}(\mathcal{C}) = L$.

The usefulness of the above theorem is that it provides a characterisation of polyhedrally complete logics purely in terms of Kripke frames. The results of [5] now follow from this theorem directly (although the original proof uses the same idea).

Corollary 4. IPC *and* BD_n *for $n \in \omega$ are polyhedrally complete.*

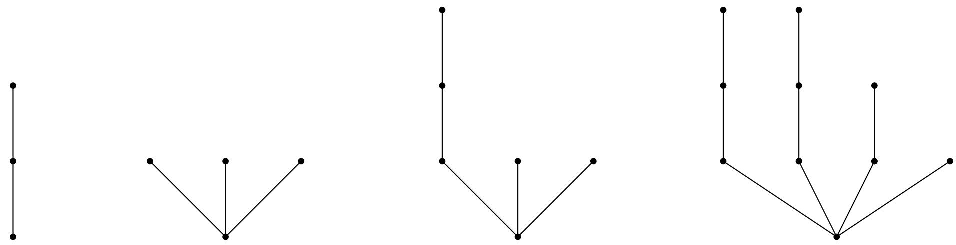


Figure 1: Some examples of starlike trees

Are there further examples of polyhedrally complete logics? The next corollary of Theorem 3 shows that we must search outside the realm of stable logics: a logic L is *stable* if the class $\mathrm{Frames}(L)$ of all rooted frames of L is closed under onto monotone images [4].

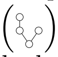
Corollary 5. *If L is polyhedrally complete, stable, and of height above 3, then $L = \mathsf{IPC}$. Therefore, BW_n, BTW_n, LC, and KC are polyhedrally incomplete.*

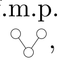
There are continuum many stable logics and all of them have the finite model property. Thus, there are continuum many logics with the f.m.p. that are not polyhedrally complete. In fact, one may wonder whether there are any polyhedrally complete logics of infinite height beyond IPC.

We provide a positive answer, and at the same time take a step towards a classification of polyhedrally complete logics. For a rooted frame F let $\chi(F)$ denote the Jankov-Fine formula of F [6, Chapter 9]. We are interested in logics axiomatised by the Jankov-Fine formulas of a certain class of frames, called 'starlike trees'.

Definition 6. A *starlike tree* is a finite tree (partial order) in which every non-root node has at most one immediate successor. See Fig. 1 for examples of starlike trees.

Theorem 7. *Let Σ be a collection of starlike trees excluding* *. Then the logic L axiomatised by IPC plus $\chi(T)$ for every $T \in \Sigma$ is polyhedrally complete if and only if it has the finite model property.*

This provides a wide class of polyhedrally complete logics. For instance, Scott's logic, $\mathsf{SL} = \mathsf{IPC} + ((\neg\neg p \to p) \to (p \vee \neg p)) \to (\neg p \vee \neg\neg p)$ is an infinite-height polyhedrally complete logic. It is well-known that $\mathsf{SL} = \mathsf{IPC} + \chi$() and that it has the f.m.p. [6, Chapter 9]. Therefore, Theorem 7 tells us that SL is polyhedrally complete.

Further, if C_n is the linear order on n elements, we have that $\mathsf{BD}_n = \chi(\mathrm{C}_{n+1})$. It is well-known that any logic extending BD_n has the f.m.p. [6, Theorem 8.85]. Therefore by Theorem 7, if Σ is a collection of starlike trees excluding , the logic axiomatised by BD_n plus $\chi(T)$ for every $T \in \Sigma$ is polyhedrally complete. This gives an infinite class of polyhedrally complete logics of each finite height.

What is the geometric interpretation of $\chi(T)$ for T a starlike tree? For this, we need a couple of definitions coming from polyhedral geometry.

Definition 8 (See [8])**.** A set of points $\{a_0, \ldots, a_d\} \subseteq \mathbb{R}^n$ is *geometrically independent* if for every $t_0, \ldots, t_d \in \mathbb{R}$ whenever:

$$\sum_{i=0}^{d} t_i a_i = 0 \qquad \text{and} \qquad \sum_{i=0}^{d} t_i = 0$$

we must have $t_0 = \cdots = t_d = 0$. The *d-simplex* on such a set is the collection of all points $x \in \mathbb{R}^n$ of the form:

$$x = \sum_{i=0}^{d} t_i a_i \qquad \text{where} \qquad \sum_{i=0}^{d} t_i = 1 \text{ and each } t_i \geq 0$$

Let $n_1, \ldots, n_k, m_1, \ldots, m_k$ be such that when the root is removed from T, we are left with a disjoint union of chains, m_i-many of height n_i for each i, and nothing else. Then if P is a polyhedron, $P \vDash \chi(T)$ expresses that for every convex, polyhedral, open neighbourhood of P, there is no way of removing a d-simplex such that what remains can be partitioned into open sets

$$C_1^1, \ldots, C_{b_1}^1, C_1^2, \ldots, C_{b_2}^2, \ldots, C_1^k, \ldots, C_{b_k}^k$$

such that C_j^i has dimension at least $a_i + d$ for every i, j.

While this geometric interpretation exists, the proof of Theorem 7 is entirely combinatorial, making use of the nerve criterion for polyhedral completeness. Indeed, the proof is based on a method of transforming a finite frame F of L (as in the statement of Theorem 7) into a frame F', mapping p-morphically onto F, with the property that $(\mathcal{N}^*)^n(F) \vDash L$ for every $n \in \mathbb{N}$.

References

[1] M. Aiello, J. van Benthem, and G. Bezhanishvili. Reasoning about space: the modal way. *J. Logic Comput.*, 13(6):889–920, 2003.

[2] J. van Benthem and G. Bezhanishvili. Modal logics of space. In *Handbook of Spatial Logics*, pages 217–298. Springer, Dordrecht, 2007.

[3] J. van Benthem, G. Bezhanishvili, and M. Gehrke. Euclidean hierarchy in modal logic. *Studia Logica*, 75(3):327–344, 2003.

[4] G. Bezhanishvili and N. Bezhanishvili. Locally finite reducts of Heyting algebras and canonical formulas. *Notre Dame J. Form. Log.*, 58(1):21–45, 2017.

[5] N. Bezhanishvili, V. Marra, D. McNeill, and A. Pedrini. Tarski's theorem on intuitionistic logic, for polyhedra. *Annals of Pure and Applied Logic*, 169(5):373–391, 2018.

[6] A. Chagrov and M. Zakharyaschev. *Modal logic*, volume 35 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1997. Oxford Science Publications.

[7] D. Gabelaia, K. Gogoladze, M. Jibladze, E. Kuznetsov, and M. Marx. Modal logic of planar polygons. Preprint submitted to Elsevier, 2018.

[8] J. R. Munkres. *Elements of Algebraic Topology.* CRC Press, 2018.