Bi-modal Logics of Mappings*

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Plain maps. Initially, we consider the maps $f : X \to Y$ between two sets. Without loss of generality, suppose that X and Y are disjoint sets. consider a Kripke frame $\mathfrak{F}_f = (W_f, R_f)$, where $W_f = X \sqcup Y$, $R_f = f$, i.e. we say that, pair of points $(x, y) \in W_f \times W_f$ is in the relation R_f, iff $f(x) = y$. The resulting Kripke frames are called Functional Frames. We say that the height of a frame $\mathfrak{F} = (W, R)$ is 2 if there exists $w, u \in W$, such that uRw and for any triple of points $(u, v, w) \in W \times W \times W$ either uRv or vRw fails. We say that a Kripke frame $\mathfrak{F} = (W, R)$ has no branching, if for any triple of points $(u, v, w) \in W \times W \times W$ either uRv or uRw fails. Irreflexive frames of height ≤ 2 are characterized by a formula $\Box\Box\bot$, the no branching property is characterized by a formula $\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)$. We show that a Kripke frame is a Functional Frame iff it is irreflexsive, non branching frame of height ≤ 2. The mentioned two formulas define the class of Functional Frames. Denote

$$K_f = K + \big(\Box\Box\bot\big) + \big(\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)\big)$$

Proposition 1. *The modal logic K_f is sound and complete with respect to the class of Functional Frames.*

We show that although the class of Functional Frames is modally definable, the subclasses of injective and surjective functional frames are not. If we extend the modal language by using four temporal operators $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\overrightarrow{\Diamond}$and $\overleftarrow{\Diamond}$, then the injective and surjective functional frames become definable. We interpret temporal operators as follows for a Kripke frame $\mathfrak{F} = (W, R)$ and $w \in W$,

1. $w \models \overrightarrow{\Box} p$ iff $\forall u \in W$, wRu implies $u \models p$.
2. $w \models \overleftarrow{\Box} p$ iff $\forall u \in W$, uRw implies $u \models p$.
3. $\overrightarrow{\Diamond} p = \neg\overrightarrow{\Box}\neg p$, $\overleftarrow{\Diamond} p = \neg\overleftarrow{\Box}\neg p$

We show that in the temporal language injective Functional Frames are determined by the formula

$$p \to \overrightarrow{\Box}\overleftarrow{\Box} p,$$

while surjective Function Frames are determined by the formula

$$\overrightarrow{\Diamond}\top \vee \overleftarrow{\Diamond}\top.$$

Order preserving maps. We consider the maps $f : \mathfrak{F}_1 \to \mathfrak{F}_2$ between Kripke frames $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$. The *Relational Functional Frame* associated with f is a bi-relational frame $f_R = (W, R, R_f)$, where $W = W_1 \sqcup W_2$, $R = R_1 \sqcup R_2$ and $R_f = f$. We say xRy if either xR_1y or xR_2y.

Note that (W, R_f) is a functional frame. In addition, the Relational Functional Frame f_R possesses the following *coherence property*: for any points $x, y \in W$, if $R_f(x) \neq \emptyset$ and $xRy \vee yRx$, then $R_f(y) \neq \emptyset$.

Proposition 2. *A bi-relational Kripke frame $\mathfrak{F} = (W, R, R_f)$ is Relational Functional Frame if and only if R_f is irreflexive, its height is less than 3, it is non branching and R_f, R have the coherence property.*

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Since we deal with bi-relational frames. We have $\Box$, $\Diamond$ for R_f and $\boxplus$, $\diamondplus$ for R in our bi-modal language. Here $\Box$, $\Diamond$ are interpreted as in Functional Frames and the $\boxplus$, $\diamondplus$ are interpreted as follows

- For any formula φ in a language, $\boxplus\varphi$ is satisfiable in $w \in W$ if φ is satisfiable in all R-successors of w.
- For any formula φ in a language, $\diamondplus\varphi$ is satisfiable in $w \in W$ if there exists an R-successor $u \in W$ of w, such that φ is satisfiable in u.

Coherence property in the Relational Functional Frames corresponds to the following formulas

$$\Diamond\top \to \boxplus\Diamond\top$$

$$\diamondplus\Diamond\top \to \Diamond\top$$

We show that the class of Relational Functional Frames is modally definable in the bi-modal language. Let K_R be defined as follows

$$K_R = K_{\Box\boxplus} + \big(\Box\Box\bot\big) + \big(\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)\big) + \big(\Diamond\top \to \boxplus\Diamond\top\big) + \big(\diamondplus\Diamond\top \to \Diamond\top\big)$$

Proposition 3. *Bi-modal logic K_R is sound and complete with respect to Relational Functional Frames.*

We show that the class of p-morphic Relational Functional Frames is also modally definable. Let the bi-modal logic of p-morphic Relational Functional Frames be denoted by K_p.

Proposition 4.

$$K_p = K_R + \big(\diamondplus\Diamond p \leftrightarrow \Diamond\diamondplus p\big)$$

We also axiomatize the bi-modal logic of order preserving Relational Functional Frames denoted by K_o. Moreover, we show the following:

Proposition 5. *K_R and K_o have the finite model property.*

Continuous maps. Now instead of relations, we equip the domain and co-domain of a Functional Frame with topological structure. Suppose $f : X_1 \to X_2$ is a map between topological spaces (X_1, τ_1) and (X_2, τ_2). Let us introduce $f_\tau = (X, \tau, R_f)$ topological structure, where $X = X_1 \sqcup X_2$, τ is a topology generated by $\tau_1 \sqcup \tau_2$, and $R_f = f$. A topological structure $f_\tau = (X, \tau, R_f)$ is called a *Topological Functional Frame* if (X, R_f) is a Functional Frame and $R_f^{-1}(X) = \{x \in X \mid \exists y \in X \text{ with } yR_f x\}$ is clopen (coherence property). Again due to existence of two, topological and function structures, we have two kinds of modal operators in our language, $\Box$, $\Diamond$ and $\boxplus$,$\diamondplus$ respectively. The operator $\Diamond$ is interpreted as f^{-1}. The operator $\boxplus$ is interpreted as topological *Interior* operator and the operator $\diamondplus$ - as topological *Closure* operator. We show that Topological Functional Frames are modally definable. The bi-modal logic of Topological Functional Frames is denoted by $S4_R$. We axiomatize this logic as follows:

$$S4_R = K_R + (\boxplus p \to p) + (\boxplus p \to \boxplus\boxplus p).$$

Proposition 6. *Bi-modal logic $S4_R$ is sound and complete with respect to Topological Functional Frames.*

Furthermore, we characterize the subclasses of continuous, open and interior Topological Functional Frames modally and axiomatize the corresponding bi-modal logics.

The following literature was used:[1], [2], [3].

References

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