An Axiomatization of the d-Logic of Planar Polygons

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Introduction. Topological semantics of modal logic starts with McKinsey and Tarski paper [7] where semantic treatment of modal diamond is provided by closure operator of a space. Every topological model generates a Closure algebra (Boolean algebra with closure operator) which is a subalgebra of closure algebra of all subsets of a model. It turns out that the set of all polygons, in particular the planar polygons, also forms a closure algebra. This idea has been explored and developed in [6]. In C-semantics based on planar polygons, modal formulas are evaluated on a subalgebra of the closure algebra generated by all planar polygons.

An alternative topological semantics also suggested in [7] and later studied in detail by number of authors [4], [5], [8] etc., is provided by derivative operator interpretation of the diamond modality. More precisely, the truth set of $\Diamond p$ in a topological model is a set of all limit points of the truth set of p. A derivative algebra is a Boolean algebra with operator which satisfies algebraic properties of topological derivative operator. The set of planar polygons generates a derivative algebra. In this paper, we give an axiomatization of the modal logic generated by the derivative algebra of planar polygons. The main result (Theorem 2) states that the logic $\mathbf{PL}_d^2$ described in the next paragraph is sound and complete d-logic of the polygonal plane.

Syntax. Fix a signature consisting of countable set $\mathbf{Prop}$ of symbols for propositions. The *propositional modal language* consists of formulas φ that are built up inductively according to the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi,$$

where p ranges over proposition symbols in $\mathbf{Prop}$. The logical symbols '$\top$' and '$\bot$', and the additional connectives such as '$\vee$', '$\rightarrow$' and '$\leftrightarrow$' and the dual modalities '$\Box$' are defined as usual.

Let $\mathbf{PL}_d^2$ be a classical normal modal logic containing the following axiom schemes:

$$\begin{array}{ll}
(I) & \Diamond\Diamond p \leftrightarrow \Diamond p \\
(II) & \Diamond\top \\
(III) & (\Diamond p \wedge \Diamond\neg p) \rightarrow \Diamond\big((p \wedge \Diamond\neg p) \vee (\neg p \wedge \Diamond p)\big) \\
(IV) & \Box\big(p \rightarrow \Box(\neg p \rightarrow \Box\neg p)\big) \\
(V) & \Box(r \rightarrow \gamma) \rightarrow \big(r \rightarrow \Diamond(\neg r \wedge \Diamond\Box p \wedge \Diamond\Box\neg p)\big)
\end{array}$$

where γ is the formula $\Diamond\Box(p \wedge q) \wedge \Diamond\Box(p \wedge \neg q) \wedge \Diamond\Box(\neg p \wedge q)$

Kripke Semantics. An adequate Kripke semantics for the logic $\mathbf{PL}_d^2$ is provided by the class of finite frames which we call crown frames with irreflexive root. Below we give the definition of these frames.

Definition 1. *A* crown frame with irreflexive root $\mathfrak{S}_n$ *is a frame* (S_n, Q_n) *such that* $S_n = \{r, s_1, \cdots, s_{2n}\}$ *and* Q_n *is defined as follows:*

$$
\begin{array}{ll}
(r, r) \notin Q_n; & \\
(r, s_i) \in Q_n & \textit{for all } s_i \in S_n; \\
(s_i, s_i) \in Q_n & \textit{for all } s_i \in S_n; \\
(s_i, s_j) \in Q_n & \textit{when } i < 2n \textit{ is even and } j = i-1, i+1; \\
(s_{2n}, s_1) \in Q_n; & \\
(s_{2n}, s_{2n-1}) \in Q_n. &
\end{array}
$$

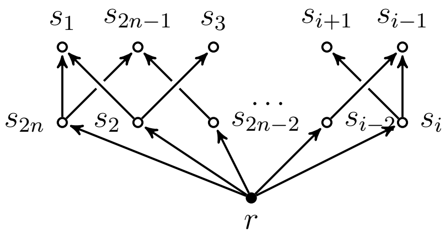

Figure 1: Crown frame

Figure 1 represents the structure of the crown frame with irreflexive root. The black bullet represents an irreflexive point and the white bullets represent reflexive points. Let **CROWN** denote the class of all crown frames with irreflexive root. We only consider Kripke models which are based on crown frames with irreflexive root. We omit the definition of satisfaction and validity of modal formula in Kripke structures. These are standard definitions and can be found in any modal logic textbook. Crown frames form the main link towards proving the topological completeness theorem essential part of which is provided by the following Kripke completeness result.

Theorem 1. $\mathbf{PL}_d^2$ *is sound and complete w.r.t. the class* ***CROWN***.

Topological Semantics. Now we define the topological semantics for our modal language. The main object of our study is the *polygonal plane* which we now proceed to define. Consider the regions of the plane obtained by the intersections of finitely many half-planes and generate the Boolean algebra using the set-theoretic operations from these regions. An arbitrary member of the obtained boolean algebra is called a *polygon* and the collection of all polygons is denoted by P_2. The structure $\mathfrak{P}_2 = (\mathbb{R}^2, P_2)$ is called the *polygonal plane*. A typical bounded member of P_2 is a finite union of (open) n-gons, line segments and points. In other words, we consider as entities not only the 2-dimensional n-gons, but also their boundaries, i.e. 'polygons' of lower dimension. It is readily checked that P_2 forms the derivative algebra.

To interpret the modal language over the polygonal plane, we allow for valuations $\nu : \textsc{Prop} \to P_2$ to range over polygons only. The valuations are extended to arbitrary modal formulas using the set-theoretic counterparts for the propositional connectives, interpreting $\Diamond$ as the topological derivative operator, and $\Box$ as its dual in the following way:

$$\begin{array}{lll} x \models p & \text{iff} & x \in \nu(p); \\ x \models \neg\varphi & \text{iff} & x \not\models \varphi; \\ x \models \varphi \vee \psi & \text{iff} & x \models \varphi \text{ or } x \models \psi; \\ x \models \Diamond\varphi & \text{iff} & x \in d(|\varphi|); \end{array}$$

Where $|p| = \{x \in \mathbb{R}^2 \mid x \models p\}$. The regions denoted by the propositional letters are specified in advance by means of a valuation, and $\vee$, $\neg$ and $\Diamond$ are interpreted as union, complement and derivative operator respectively. We skip the standard notions from general topology and definitions of satisfaction at a point, validity in the topological model etc. The reader is referred to [3], [1] for these definitions. To give the proof idea of the main theorem we need a definition of maps which preserve modal formulas i.e. maps which are alike to p-morphisms, but when one structure is a topological space and the other one—a Kripke frame.

Definition 2 ([2])**.** *A map $f : X \to W$ where (X, τ) is a topological space and (W, R) is a transitive Kripke frame, is called a d-morphism if the following properties are satisfied:*

(i) For each open $U \in \tau$ it holds that $R\big(f(U)\big) \subseteq f(U)$;

(ii) For each $V \subseteq W$ such that $R(V) \subseteq V$ it holds that $f^{-1}(V) \in \tau$;

(iii) For each irreflexive point $w \in W$ it holds that $f^{-1}(w)$ is discrete space w.r.t. subspace topology;

(iv) For each reflexive point $w \in W$ it holds that $f^{-1}(w) \subseteq d(f^{-1}(w))$.

It is a well known fact that d-morphisms preserve validity of modal formulas.

Proposition 1. *Every rooted crown frame $\mathfrak{F}$ is a d-morphic image of a polygonal plane $\mathfrak{P}_2$ in such a way that the preimage of an arbitrary point $w \in \mathfrak{F}$ belongs to the algebra of planar polygons P_2.*

Now we are ready to state the main result of our paper.

Theorem 2. $\mathbf{PL}_d^2$ *is sound and complete logic of polygonal plane $\mathfrak{P}_2$.*

Proof. (Sketch of Completeness). By Theorem 1, for an arbitrary formula φ which is not a theorem of $\mathbf{PL}_d^2$, there exists a rooted crown frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. By Proposition 1, φ is falsified on $\mathfrak{P}_2$. □

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