Rules of Inference

Lecture 2

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Rosalie Iemhoff
Utrecht University, The Netherlands

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Today

- Examples
- Bases
- Approximations
- Projective formulas

Dfn Given a set of rule schemes $\mathcal{R}$, $\vdash^{\mathcal{R}}$ is the smallest consequence relation that extends $\vdash$ and which rules contain Ru($\mathcal{R}$).

For a rule R, $\vdash^{R}$ is short for $\vdash^{\mathcal{R}}$, where $\mathcal{R}$ consists of the rule scheme (R, Sub), for Sub being the set of all substitutions.

Dfn Γ/Δ is *derivable* in L iff $\Gamma \vdash_{\mathsf{L}} \Delta$.

Dfn $R = \Gamma/\Delta$ is *admissible* in L ($\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta$) iff $\mathsf{Thm}(\vdash_{\mathsf{L}}) = \mathsf{Thm}(\vdash_{\mathsf{L}}^{R})$.

Thm For single-conclusion consequence relations:

$$\Gamma \mathrel{|\!\sim}_{\mathsf{L}} A \text{ iff for all substitutions } \sigma\text{: } \vdash_{\mathsf{L}} \bigwedge \sigma\Gamma \text{ implies } \vdash_{\mathsf{L}} \sigma A.$$

Thm For multi-conclusion c.r.'s with the disjunction property:

$$\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta \text{ iff for all substitutions } \sigma\text{: } \vdash_{\mathsf{L}} \bigwedge \sigma\Gamma \text{ implies } \vdash_{\mathsf{L}} \sigma A \text{ for some } A \in \Delta.$$

Examples

Thm Classical propositional logic CPC is structurally complete, i.e. all admissible rules of $\vdash_{\mathsf{CPC}}$ are (strongly) derivable.

Thm The *Harrop* or *Kreisel-Putnam Rule*

$$\frac{\neg A \to B \vee C}{(\neg A \to B) \vee (\neg A \to C)} \text{ HR}$$

is admissible but not derivable in IPC, as

$$(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$$

is not derivable in IPC. The same holds for Heyting Arithmetic.

Thm (Prucnal '79) HR is admissible in any intermediate logic.

Thm The *disjunctive Harrop Rule*

$$\frac{\{\neg A \to B \vee C\}}{\{(\neg A \to B), (\neg A \to C)\}} \text{ HR}$$

is admissible in intermediate logics with the disjunction property.

Thm (Buss & Mints & Pudlak '01)
HR does not shorten proofs more than polynomially.

Thm (Prucnal '79) The *Harrop* or *Kreisel-Putnam Rule*

$$\frac{\neg A \to B \vee C}{(\neg A \to B) \vee (\neg A \to C)}\ \mathrm{HR}$$

is admissible in any intermediate logic.

Prf If $\vdash_{\mathsf{L}} \neg A \to B \vee C$, then $\vdash_{\mathsf{L}} \neg\sigma A \to \sigma B \vee \sigma C$, where $\sigma = \sigma_I^{\neg A}$ for some valuation v_I that satisfies $\neg A$ (if $\neg A$ is inconsistent, the statement is trivial).

As $\vdash_{\mathsf{CPC}} \neg\sigma A$, also $\vdash_{\mathsf{L}} \neg\sigma A$ by Glivenko's theorem. Hence $\vdash_{\mathsf{L}} \sigma B \vee \sigma C$. Therefore $\vdash_{\mathsf{L}} (\neg A \to B) \vee (\neg A \to C)$. □

Thm (Minari & Wroński '88) For A a Harrop formula, the rule

$$\frac{A \to B \vee C}{(A \to B) \vee (A \to C)}$$

is admissible in any intermediate logic.

Decidability

Thm (Rybakov '80's) $\mathrel{|\!\!\sim}_{\mathsf{IPC}}$, $\mathrel{|\!\!\sim}_{\mathsf{K4}}$, $\mathrel{|\!\!\sim}_{\mathsf{S4}}$, $\mathrel{|\!\!\sim}_{\mathsf{GL}}$ are decidable.

Thm (Chagrov '92) There are decidable logics in which admissibility is undecidable.

Thm (Rybakov & Odintsov & Babenyshev '00's)
Admissibility is decidable in many modal and temporal logics.

Thm (Jeřábek '07)
In IPC and many transitive modal logics admissibility is coNEXP-complete.

Bases

Note If $A \mathrel{|\!\sim}_{\mathsf{L}} B$, then $A \wedge C \mathrel{|\!\sim}_{\mathsf{L}} B \wedge C$.

To describe all admissible rules of a logic the notion of basis is used.

Dfn A set of rules $\mathcal{R}$ *derives* a rule Γ/Δ in L iff $\Gamma \vdash_{\mathsf{L}}^{\mathcal{R}} \Delta$.

Dfn $\mathcal{R}$ is a *basis* for the admissible rules of L iff the rules in $\mathcal{R}$ are admissible in L and all admissible rules of L are derivable from $\mathcal{R}$ in L:

$$\mathrel{|\!\sim}_{\mathsf{L}} = \vdash_{\mathsf{L}}^{\mathcal{R}} .$$

Dfn A basis is *independent* if no proper subset of it is a basis. It is *weakly independent* if no finite subset of it is a basis.

Thm (Rybakov 80's)
There is no finite basis for the admissible rules of IPC.

In the description of bases it is convenient to use sequents instead of formulas.

Dfn A *sequent* is of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas. Its interpretation $I(\Gamma \Rightarrow \Delta)$ is $\bigwedge \Gamma \to \bigvee \Delta$.

With a formula A the sequent $\Rightarrow A$ is associated.

We sometimes write $\vdash S$ instead of $\vdash I(S)$.

For a set of sequents $\mathcal{S}$, $I(\mathcal{S})$ denotes $\bigwedge_{S \in \mathcal{S}} I(S)$.

Dfn An implication $A \to B$ is *atomic* if A and B are atoms. A sequent $(\Gamma \Rightarrow \Delta)$ is *irreducible* if Δ consists of atoms and Γ of atoms and atomic implications.

An implicational formula $\bigwedge \Gamma \to \bigvee \Delta$ is *irreducible* if $\Gamma \Rightarrow \Delta$ is.

In IPC:

Formulas	$A \vee B \mathrel{|\!\!\sim} \{A, B\}$	$\bigvee \Delta \mathrel{|\!\!\sim} \Delta$
Sequents	$\Rightarrow A, B \mathrel{|\!\!\sim} \{\Rightarrow A, \Rightarrow B\}$	$\Rightarrow \Delta \mathrel{|\!\!\sim} \{\Rightarrow D \mid D \in \Delta\}$
HR	$\neg A \Rightarrow \Delta \mathrel{|\!\!\sim} \{\neg A \Rightarrow D \mid D \in \Delta\}$	
	$A \to B \Rightarrow \Delta \mathrel{|\!\!\sim} \{A \to B \Rightarrow D \mid D \in \Delta\} \cup \{A \to B \Rightarrow A\}$	
Visser rules	$\Gamma \Rightarrow \Delta \mathrel{|\!\!\sim} \{\Gamma \Rightarrow D \mid D \in \Delta \cup \Gamma^a\}$ (Γ implications only).	

Γ^a consists of the A such that $(A \to B) \in \Gamma$ for some B.

Thm (Iemhoff '01, Roziére '92)
The Visser rules are a basis for the multi-conclusion admissible rules of IPC.

Dfn The single-conclusion Visser rules: (Γ implications only)

$$(\bigwedge \Gamma \to \bigvee \Delta) \vee A \ / \ \bigvee \{\bigwedge \Gamma \to D \mid D \in \Delta \cup \Gamma^a\} \vee A.$$

Dfn Intermediate logics:

KC	$\neg A \vee \neg\neg A$	a maximal node
LC	$(A \to B) \vee (B \to A)$	linear

Thm (Iemhoff '05)
The single-conclusion Visser rules are a basis for the admissible rules in any intermediate logic in which they are admissible.

Thm The single-conclusion Visser rules are a basis for the admissible rules of KC.

Thm The single-conclusion Visser rules are derivable in LC. Hence LC is structurally complete.

Thm (Goudsmit & Iemhoff '12) The $(n+1)$-th Visser rule is a basis for the n-th Gabbay-deJongh logic.

Dfn Given a formula A and set of atoms I, valuation v_I and substitution σ_I^A are defined as

$$v_I(p) \equiv_{dfn} \begin{cases} 1 & \text{if } p \in I \\ 0 & \text{if } p \notin I \end{cases} \qquad \sigma_I^A(p) \equiv_{dfn} \begin{cases} A \to p & \text{if } p \in I \\ A \wedge p & \text{if } p \notin I. \end{cases}$$

Thm If S contains an atom, then for $I(S) = A$, $A \mathrel{|\!\sim} B \;\Leftrightarrow\; A \vdash B$.

Prf Choose an atom p in S. Define σ to be $\sigma_\emptyset^A$ if p is in the antecedent of S, and $\sigma_{\{p\}}^A$ otherwise.

$\vdash \sigma A$ and $A \vdash \sigma(B) \leftrightarrow B$ for all B. Thus $A \mathrel{|\!\sim} B$ implies $A \vdash B$. $\square$.

Note In many modal logics, any nonderivable admissible rule formulated via sequents has to have a premiss that does not contain atoms.

Dfn The modal Visser rules:

$$\frac{\Box\Gamma \Rightarrow \Box\Delta}{\{\boxdot\Gamma \Rightarrow D \mid D \in \Delta\}}\ \mathsf{V}^\bullet \qquad \frac{\{\Box\Gamma \equiv \Gamma \Rightarrow D \mid D \in \Delta\}}{\{\boxdot\Gamma \Rightarrow D \mid D \in \Delta\}}\ \mathsf{V}^\circ$$

($\boxdot A$ denotes $A \wedge \Box A$ and $\Box\Gamma \equiv \Gamma$ denotes $\{A \leftrightarrow \Box A \mid A \in \Gamma\}$.)

Thm (Jeřábek '05)
The irreflexive Visser rules are a basis in any extension of GL in which they are admissible. Similarly for the reflexive Visser rules and S4, and for their combination and K4.

Thm (Babenyshev & Rybakov '10)
Explicit bases for temporal modal logics.

Thm (Mints '76)
In IPC, all nonderivable admissible rules contain $\vee$ and $\to$.

Thm (Prucnal '83)
$\mathrm{IPC}_{\to}$ is structurally complete, as is $\mathrm{IPC}_{\to,\wedge}$.

Thm (Minari & Wroński '88)
$\mathrm{IPC}_{\to,\neg,\wedge}$ is structurally complete.

Thm (Cintula & Metcalfe '10)
$\mathrm{IPC}_{\to,\neg}$ is not structurally complete. The Wroński rules are a basis for its admissible rules:

$$\frac{(p_1 \to (p_2 \to \ldots (p_n \to \bot) \ldots)}{\{\neg\neg p_i \to p_i \mid i = 1, \ldots, n\}}$$

Thm (Odintsov & Rybakov '12)
Johanssons' minimal logic has finitary unification and admissibility is decidable.

Thm (Jeřábek '09)
The admissible rules of Łukasiewicz logic have no finite basis, but a nice infinite basis exists.

Approximations

Thm In many intermediate and modal logics, there is for every formula A a finite set of irreducible formulas Π_A such that

$$\bigvee \Pi_A \vdash A \mathrel{|\!\sim} \Pi_A,$$

and for all $B \in \Pi_A$ and all C, $B \mathrel{|\!\sim} C \;\Leftrightarrow\; B \vdash C$.

Cor If also $A \vdash^{\mathcal{R}} \Pi_A$ for some set of admissible rules $\mathcal{R}$, then $\mathcal{R}$ is a basis.

Prf $A \mathrel{|\!\sim} C$ implies that $B \vdash C$ for all $B \in \Pi_A$. Hence $A \vdash^{\mathcal{R}} C$. □

Dfn Π_A is an *(irreducible) projective approximation* of A.

Dfn $A \Vdash B$ if there is a σ which is the identity on the atoms in A such that $A \vdash \sigma B$. $A \Vdash\!\!\sim B$ if every unifier of A can be extended to a unifier of B.

Thm Given a sequent S there is a set $\mathcal{G}$ of irreducible sequents such that

$$I(S) \Vdash \bigwedge I(\mathcal{G}) \vdash I(S).$$

Prf (I) Apply the invertible logical rules of LJ as long as possible:

For example, $\Gamma, A \wedge B \Rightarrow \Delta$ is replaced by $\Gamma, A, B \Rightarrow \Delta$.

(II) Introduce atoms for the composite formulas in S:

For example, $\Gamma, A \to B \Rightarrow \Delta$ is replaced by

$$(\Gamma, p \to q \Rightarrow \Delta) \;\; (p \Rightarrow A) \;\; (B \Rightarrow q).$$

Apply (I) and (II) as long as possible.

Dfn Given a formula A and set of atoms I, valuation v_I and substitution σ_I^A are defined as

$$v_I(p) \equiv_{dfn} \begin{cases} 1 & \text{if } p \in I \\ 0 & \text{if } p \notin I \end{cases} \qquad \sigma_I^A(p) \equiv_{dfn} \begin{cases} A \to p & \text{if } p \in I \\ A \wedge p & \text{if } p \notin I. \end{cases}$$

Note $A \vdash \sigma_I^A(B) \leftrightarrow B$ for all B and I.

Note If $\vdash \sigma_I^A(A)$, then $A \mathrel{|\!\sim} B \;\Leftrightarrow\; A \vdash B$ for all B.

Projective formulas

Dfn (Ghilardi) A formula A is *projective* in L if for some substitution σ and all atoms p:

$$\vdash_{\mathsf{L}} \sigma A \quad A \vdash_{\mathsf{L}} p \leftrightarrow \sigma(p).$$

σ is the *projective unifier (pu)* of A.

Thm If A is projective and $\vdash$ has the disjunction property, then for all Δ:

$$A \mathrel{|\!\sim}_{\mathsf{L}} \Delta \;\Leftrightarrow\; \exists B \in \Delta \; A \vdash_{\mathsf{L}} B.$$

Cor If all unifiable formulas are projective in L, then all nonpassive rules are derivable.

Ex For $I = \{p\}$, σ_I^p is a pu of p. For $I = \emptyset$, $\sigma_I^{\neg p}$ is a pu of $\neg p$.

Dfn $\sum K_i$ denotes the disjoint union of the models $K_1, \ldots, K_n$.

Dfn K' denotes the extension of model K with one node at which no atoms are forced and that is below all nodes in K.

Dfn Two rooted models on the same frame are *variants* of each other when their valuation differs at most at the root.

Dfn A class of Kripke models $\mathcal{K}$ has the *extension property (EP)* if for all $K_1, \ldots, K_n \in \mathcal{K}$ there is a variant of $(\sum K_i)'$ in $\mathcal{K}$.

Dfn A formula A has the *extension property* if it is complete with respect to a class of models with the extension property.

Thm (Ghilardi) In IPC, A is projective iff A has EP.

Ex In IPC, p and $\neg p$ are projective and $p \vee q$ is not.

Similar techniques apply to modal logics.

Method of proof

Thm If there is a set of admissible rules $\mathcal{R}$ such that for every formula A there is a finite set of projective formulas Π_A such that

$$\bigvee \Pi_A \vdash_{\mathsf{L}} A \vdash_{\mathsf{L}}^{\mathcal{R}} \Pi_A,$$

then $\mathcal{R}$ is a basis for the admissible rules of L.

Thm In the following logics there exists for every formula A a finite set of projective formulas Π_A such that

- in IPC: $\bigvee \Pi_A \vdash A \vdash^{\mathsf{V}} \Pi_A$;
- in S4: $\bigvee \Pi_A \vdash A \vdash^{\mathsf{V}^\circ} \Pi_A$;
- in GL: $\bigvee \Pi_A \vdash A \vdash^{\mathsf{V}^\bullet} \Pi_A$;
- in $\mathsf{CPC}_{\neg,\to}$: $\bigvee \Pi_A \vdash A \vdash^{\mathsf{W}} \Pi_A$;
- ...

(Jeřábek) In Ł: similar but the formulas are not projective.

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