Rules of Inference

Lecture 1

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Questions

Given a theorem, what are the proofs that prove it?

Given a logic, what are the systems of inference that describe it?

Independent of the representation of a logic, in most cases there is a notion of inference.

Given a rule of inference, is it derivable in the logic or can it be added without extending the set of theorems? Is it *admissible*?

These questions can be studied using proof theory, model theory, algebraic logic . . . Close connection unification theory.

Ex

- $B \to A(x)/B \to \forall y A(y)$ (x not free in B) is admissible in classical predicate logic.
- Con(ZF)$/\bot$ is admissible but not derivable in ZF.
- $\Box A/A$ is admissible in many modal logics.
- Markov's Rule $\neg\neg\exists x A(x)/\exists x A(x)$ for $A \in \Delta_0$ is admissible in Heyting Arithmetic.
- Cut is admissible in Gentzen's sequent calculus LK $- \mathrm{Cut}$.
- The Density Rule $(A \to P) \vee (P \to B)/A \to B$ for atomic P is admissible in first-order Gödel logic.

Today

- Introduction
- Consequence relations
- Rules
- Decidability

Introduction

Ex Let T be a theory in the propositional language consisting of propositional variables and $\to$ given by (all substitution instances of)

$$\frac{}{A \to A} \quad \frac{A \quad A \to B}{B} .$$

The rule

$$\frac{B \to A}{A \to B}$$

is admissible in T, but not derivable:

$$\vdash A \to B \Rightarrow \vdash B \to A$$

$$A \to B \nvdash B \to A.$$

Ex The sequent calculus LK for classical predicate logic CQC contains the Cut Rule

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Cut$$

(Γ and Δ are finite sequences of formulas.) LK^-: LK without *Cut*.

Thm Every theorem of CQC has a proof in LK without *Cut*.

In other words: *Cut* is admissible in LK^-.

Disjunction properties are exampels of multi-conclusion rules:

Ex If $\vdash_{\mathsf{IPC}} A \vee B$ then $\vdash_{\mathsf{IPC}} A$ or $\vdash_{\mathsf{IPC}} B$.

Ex If $\vdash_{\mathsf{K4}} \Box A \vee \Box B$ then $\vdash_{\mathsf{K4}} A$ or $\vdash_{\mathsf{K4}} B$.

Other multi-conclusion rules:

Ex (Williamson '92) If $\vdash_{\mathsf{KT}} A \to \Box A$ then $\vdash_{\mathsf{KT}} A$ or $\vdash_{\mathsf{KT}} \neg A$.

Logics can be given in a variety of ways.

What does it mean to extend a logic by a rule?

What does it mean to extend a logic by a multi-conclusion rule?

Consequence Relations

Setting

Consequence relations form a convenient framework to study rules.

They capture in great generality what it means to infer something.

Origins: Tarski (1935), Polish School.

Dfn We consider logics in a certain language $\mathcal{L}$ together with a notion of substitution for the formulas $\mathcal{F}_{\mathcal{L}}$ in $\mathcal{L}$.

Propositional logics: $\mathcal{L}_p$ consists of propositional variables (atoms), constants $\top, \bot$ and connectives $\wedge, \vee, \to, \neg$. A substitution σ is a map on $\mathcal{F}_{\mathcal{L}}$ that commutes with the connectives. It is uniquely characterized by its behavior on atoms.

Modal logics: $\mathcal{L}_m$ is $\mathcal{L}_p$ extended by the modal operators and substitutions commute with the connectives and the operators.

Predicate logics: $\mathcal{L}_f$ is defined as usual, with predicates, functions, variables, the connectives, $\top$, $\bot$, and the quantifiers.

Substructural logics: $\mathcal{L}_s$ is as propositional logic except that the connectives may be different, such as ! or $\otimes$.

Dfn A *finitary multi-conclusion consequence relation (m.c.r.)* $\vdash$ is a relation on finite sets of formulas in $\mathcal{F}_{\mathcal{L}}$ that satisfies

Reflexivity	$\Gamma \vdash \Delta$ if $\Gamma \cap \Delta \neq \emptyset$;
Monotonicity	$\Gamma \vdash \Delta$ implies $\Gamma, \Pi \vdash \Delta, \Sigma$;
Transitivity	$\Gamma \vdash A, \Delta$ and $\Pi, A \vdash \Sigma$ implies $\Gamma, \Pi \vdash \Delta, \Sigma$.

It is *structural* if

Structurality	$\Gamma \vdash \Delta$ implies $\sigma\Gamma \vdash \sigma\Delta$ for all substitutions σ.

$\Gamma, A \vdash \Delta, B$ is short for $\Gamma \cup \{A\} \vdash \Delta \cup \{B\}$ and $\vdash \Delta$ for $\emptyset \vdash \Delta$.

A *finitary single-conclusion consequence relation (s.c.r.)* $\vdash$ is a relation between finite sets of formulas and formulas that satisfies

Reflexivity	$A \vdash A$;
Monotonicity	$\Gamma \vdash A$ implies $\Gamma, \Pi \vdash A$;
Transitivity	$\Gamma \vdash A$ and $\Pi, A \vdash B$ implies $\Gamma, \Pi \vdash B$.

Dfn The *theorems* of $\vdash$ are $\mathrm{Th}(\vdash) \equiv_{dfn} \{A \mid \; \vdash A \text{ holds}\}$.

The *multi-conclusion theorems* of $\vdash$ are $\mathrm{Thm}(\vdash) \equiv_{dfn} \{\Delta \mid \; \vdash \Delta \text{ holds}\}$.

Ex The minimal consequence relation: $\Gamma \vDash^{m} \Delta \equiv_{dfn} \Gamma \cap \Delta \neq \emptyset$.

Ex Given a m.c.r. $\vdash$, its single-conclusion fragment $\vdash_s$ is defined as

$$\Gamma \vdash_s A \equiv_{dfn} \Gamma \vdash A.$$

Ex Any s.c.r. $\vdash$ has a natural multi-conclusion analogue:

$$\Gamma \vdash^{\mathrm{min}} \Delta \;\Leftrightarrow\; \exists A \in \Delta \; \Gamma \vdash A$$

Dfn A consequence relation $\vdash$ *covers* a logic L if $\mathrm{Th}(\vdash)$ consists of the theorems of L.

Examples of single-conclusion consequence relations

Ex The following two consequence relations cover Th(CPC):

$$\Gamma \vdash A \equiv_{dfn} (\bigwedge \Gamma \to A) \text{ is a theorem of CPC;}$$
$$\Gamma \vdash A \equiv_{dfn} A \in \Gamma \text{ or } A \text{ is a theorem of CPC.}$$

Ex A s.c.r. on $\mathcal{F}_{\mathcal{L}_m}$ that covers the modal logic K:

$$\Gamma \vdash A \equiv_{dfn} A \text{ holds in all Kripke models in which all formulas in } \Gamma \text{ hold.}$$

Dfn Given a logic L that contains $\wedge, \vee, \to$ the s.c.r. and m.c.r. $\vdash_{\mathsf{L}}$ are

$$\Gamma \vdash_{\mathsf{L}} A \equiv_{dfn} (\bigwedge \Gamma \to A) \text{ is a theorem of L}$$
$$\Gamma \mathrel{\dot{\vdash}}_{\mathsf{L}} \Delta \equiv_{dfn} (\bigwedge \Gamma \to \bigvee \Delta) \text{ is a theorem of L}$$

Note For many logics L, $\vdash_{\mathsf{L}}$ and $\dot{\vdash}_{\mathsf{L}}$ are structural consequence relations.

Consequence relations and rules

Logics are given by consequence relations.

What does it mean to extend a logic by a rule?

What does it mean to extend a logic by a multi-conclusion rule?

Rules

Dfn A *(single-conclusion) rule* is a pair of a finite set of formulas Γ and a formula A, denoted as Γ/A. A *multi-conclusion rule* is a pair of finite sets of formulas denoted as Γ/Δ. Alternative notation: $\frac{\Gamma}{A}$ and $\frac{\Gamma}{\Delta}$.

$\mathsf{Ru}(\vdash)$ consists of the rules Γ/Δ for which $\Gamma \vdash \Delta$. Similar for s.c.r.

For a set of rules $\mathcal{R}$, $\sigma\mathcal{R} \equiv_{dfn} \{\sigma R \mid R \in \mathcal{R}\}$, where $\sigma(\Gamma/\Delta) = \sigma\Gamma/\sigma\Delta$.

Because rules are often schematic and come with side conditions, the notion of rule is generalized to rule scheme.

Dfn A *rule scheme* is a pair (R, S), where R is a rule, and S a set of substitutions. For every $\sigma \in S$, σR is an *instance* of the rule scheme. A *structural* rule is a rule scheme where the set of substitutions is maximal.

Given a set of rules schemes $\mathcal{R}$:

$$\mathsf{Ru}(\mathcal{R}) \equiv_{dfn} \{\sigma R \mid \exists S : (R, S) \in \mathcal{R} \text{ and } \sigma \in S\}.$$

Examples of rule schemes

Ex

- $\exists x A(x, fx)/\exists x \forall y A(x, y)$ (f fresh) is a *Skolem Rule* in predicate logic.
- $(\Box A/A, S)$, where S is the set of all subsitutions in $\mathcal{L}_m$, is a structural rule that is admissible but nonderivable in many modal logics. $\Box(p \to q)/p \to q$ is an instance of the rule.
- Markov's Scheme $(\neg\neg\exists x P(x)/\exists x P(x), S)$, where S consists of those substitutions on $\mathcal{L}_f$ that map atom $P(x)$ to a formula in Δ_0, is admissible in Heyting Arithmetic.
- The Cut Scheme $\{(\Gamma \Rightarrow A, \Delta), (A, \Gamma \Rightarrow \Delta)\}/(\Gamma \Rightarrow \Delta), S)$, where S is the set of all subsitutions in $\mathcal{L}_f$, is an admissible structural rule scheme in Gentzen's sequent calculus $\mathsf{LK} - \mathrm{Cut}$.

Dfn Given a set of rule schemes $\mathcal{R}$, $\vdash^{\mathcal{R}}$ is the smallest consequence relation that extends $\vdash$ and which rules contain $\mathrm{Ru}(\mathcal{R})$.

For a rule R, $\vdash^{R}$ is short for $\vdash^{\mathcal{R}}$, where $\mathcal{R}$ consists of the rule scheme (R, Sub), for Sub being the set of all substitutions.

Dfn Given a set of rule schemes $\mathcal{R}$, a sequence of formulas $A_1, \ldots, A_n$ is a *derivation* of Γ/Δ in $\mathcal{R}$ if $A_n \in \Delta$ and for all $A_i \notin \Gamma$ there are $i_1, \ldots, i_m < i$ such that $A_{i_1}, \ldots, A_{i_m}/A_i$ belongs to $\mathrm{Ru}(\mathcal{R})$.

Thm For a s.c.r. $\vdash$ and set of s.c. rule schemes $\mathcal{R}$: $\Gamma \vdash^{\mathcal{R}} A$ iff there is a derivation of Γ/A in $\mathrm{Ru}(\vdash) \cup \mathrm{Ru}(\mathcal{R})$.

Dfn A m.c.r. $\vdash$ is *saturated* if $\Gamma \vdash \Delta$ implies $\Gamma \vdash A$ for some $A \in \Delta$.

Thm For a saturated m.c.r. $\vdash$ and set of m.c. rule schemes $\mathcal{R}$: $\Gamma \vdash^{\mathcal{R}} \Delta$ iff there is a derivation of Γ/Δ in $\mathrm{Ru}(\vdash) \cup \mathrm{Ru}(\mathcal{R})$.

Dfn For a logic given by a c.r. $\vdash$, the *extension* of it by a set of rules (schemes) $\mathcal{R}$ is $\vdash^{\mathcal{R}}$.

Ex Mendelson's Hilbert style system for $\mathsf{CPC}_{\neg,\to}$ given as a set of structural rules $\mathcal{R}$: rule Modus Ponens and axioms

$$A \to (B \to A) \quad (A \to (B \to C)) \to ((A \to B) \to (A \to C))$$
$$(\neg A \to \neg B) \to ((\neg A \to B) \to A).$$

Is there a smaller consequence relation that covers $\mathsf{CPC}_{\neg,\to}$? Yes,

$$\{\Gamma \vdash' A \mid \ A \in \Gamma \text{ or } A \text{ holds in } \mathsf{CPC}_{\neg,\to}\}.$$

Is there an extension of $\vdash$ that also covers $\mathsf{CPC}_{\neg,\to}$? We will see.

Dfn Γ/Δ is *derivable* in L iff $\Gamma \vdash_{\mathsf{L}} \Delta$.

Dfn Γ/Δ is *strongly derivable* in L iff $\vdash_{\mathsf{L}} \bigwedge \Gamma \to \bigvee \Delta$.

Dfn $R = \Gamma/\Delta$ is *admissible* in L ($\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$) iff $\mathrm{Thm}(\vdash_{\mathsf{L}}) = \mathrm{Thm}(\vdash_{\mathsf{L}}^{R})$.

Dfn $R = \Gamma/A$ is *admissible* in L ($\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} A$) iff $\mathrm{Th}(\vdash_{\mathsf{L}}) = \mathrm{Th}(\vdash_{\mathsf{L}}^{R})$.

Dfn $\vdash_{\mathsf{L}}$ is *structurally complete* if all admissible rules of $\vdash_{\mathsf{L}}$ are derivable.

Ex $\bot/A$ is admissible in any consistent logic, but not always derivable.

Ex $\Box A/A$ is admissible in many modal logics, such as S4 and GL.

Dfn $\vdash$ has the disjunction property iff $\vdash \Delta$ implies $\vdash A$ for some $A \in \Delta$.

Standard consequence relations

Note Given a logic L, the smallest consequence relation that covers L is

$$\{\Gamma \vdash^{m}_{\mathsf{L}} A \mid \ A \in \mathrm{Th}(\mathsf{L}) \text{ or } A \in \Gamma\}.$$

Thm For single-conclusion consequence relations:

$$\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} A \text{ iff for all substitutions } \sigma\text{: } \vdash_{\mathsf{L}} \bigwedge \sigma\Gamma \text{ implies } \vdash_{\mathsf{L}} \sigma A.$$

Thm For multi-conclusion c.r.'s with the disjunction property:

$$\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta \text{ iff for all substitutions } \sigma\text{: } \vdash_{\mathsf{L}} \bigwedge \sigma\Gamma \text{ implies } \vdash_{\mathsf{L}} \sigma A \text{ for some } A \in \Delta.$$

Note L has the disjunction property iff $A \vee B \mathrel{|\!\!\sim}_{\mathsf{L}} \{A, B\}$.

Thm For c.r. with the disjunction property admissibility depends only on the theorems of the c.r: if $\mathrm{Th}(\vdash_1) = \mathrm{Th}(\vdash_2)$, then $\mathrel{|\!\!\sim}_1 = \mathrel{|\!\!\sim}_2$.

Note $\mathrel{|\!\!\sim}_{\mathsf{L}}$ is the largest consequence relation that covers L.

Thm For all logics L:

$$\vdash^{m}_{\mathsf{L}} \subseteq \vdash_{\mathsf{L}} \subseteq \mathrel{|\!\!\sim}_{\mathsf{L}} \qquad \mathrm{Th}(\vdash^{m}_{\mathsf{L}}) = \mathrm{Th}(\vdash_{\mathsf{L}}) = \mathrm{Th}(\mathrel{|\!\!\sim}_{\mathsf{L}}).$$

Dfn A quasi equation $\{s_i = t_i \mid i = 1, \ldots, n\} \Rightarrow s = t$ is *admissible* in a class of algebras $\mathcal{K}$ if it holds in the free algebra of $\mathcal{K}$ on ω generators.

Thm Classical propositional logic CPC is structurally complete, i.e. all admissible rules of $\vdash_{\mathsf{CPC}}$ are (strongly) derivable.

Prf If A/B is admissible, then for all ground substitutions σ from $\mathcal{F}_{\mathcal{L}_p}$ to $\{\top, \bot\}$: if $\vdash_{\mathsf{CPC}} \sigma A$, then $\vdash_{\mathsf{CPC}} \sigma B$.

Thus $A \to B$ is true under all valuations. Hence $\vdash_{\mathsf{CPC}} A \to B$. □

Dfn Given a formula A and set of atoms I, valuation v_I and substitution σ_I^A are defined as

$$v_I(p) \equiv_{dfn} \begin{cases} \top & \text{if } p \in I \\ \bot & \text{if } p \notin I \end{cases} \qquad \sigma_I^A(p) \equiv_{dfn} \begin{cases} A \to p & \text{if } p \in I \\ A \wedge p & \text{if } p \notin I. \end{cases}$$

Thm If $v_I(A) = \top$, then $\vdash_{\mathsf{CPC}} \sigma_I^A(A)$.

Prf Write σ for σ_I^A. $\vdash_{\mathsf{CPC}} A \to (\sigma(p) \leftrightarrow p)$, thus $\vdash_{\mathsf{CPC}} A \to \sigma(A)$. $\vdash_{\mathsf{CPC}} \neg A \to (\sigma(p) \leftrightarrow v_I(p))$, thus $\vdash_{\mathsf{CPC}} \neg A \to \sigma(A)$. Hence $\vdash_{\mathsf{CPC}} \sigma(A)$.
□

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